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Boundary one-point function, Casimir energy and boundary state formalism in $D+1$ dimensional QFT

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Abstract

We consider quantum field theories with boundary on a codimension one hyperplane. Using $1+1$ dimensional examples, we clarify the relation between three parameters characterising one-point functions, finite size corrections to the ground state energy and the singularity structure of scattering amplitudes, respectively. We then develop the formalism of boundary states in general $D+1$ spacetime dimensions and relate the cluster expansion of the boundary state to the correlation functions using reduction formulae. This allows us to derive the cluster expansion in terms of the boundary scattering amplitudes, and to give a derivation of the conjectured relations between the parameters in $1+1$ dimensions, and their generalization to $D+1$ dimensions. We use these results to express the large volume asymptotics of the Casimir effect in terms of the one-point functions or alternatively the singularity structure of the one-particle reflection factor, and for the case of vanishing one-particle couplings we give a complete proof of our previous result for the leading behaviour.

1 Introduction

In this paper we treat quantum field theories with a boundary, which is supposed to be a hyperplane (of codimension 1) in flat spacetime. We also treat finite size effects when two such boundaries are parallel to each other and separated by a distance L (a.k.a. the Casimir effect in a planar situation).

Such theories have been extensively studied in $1+1$ dimensions since the seminal paper of Ghoshal and Zamolodchikov [1], which introduced some fundamental ideas into the subject. They defined and characterized the so-called boundary state in integrable field theories[1], which makes possible the description of boundary phenomena using the Hamiltonian formalism of the bulk theory. Using this concept, they set up the necessary notions to study integrable

[1]In Section 3 we give a definition for the boundary state in any boundary quantum field theory in arbitrary number of spacetime dimensions, but the details are not necessary for the time being.

boundary QFTs by extending the ideas of analytic S matrix theory, factorization and bootstrap already well-known in the bulk situation.

Boundary field theories in $1+1$ dimensions are relevant to numerous condensed matter phenomena (the most prominent examples are the so-called “impurity” problems), to nonperturbative aspects of string theory (“branes”) and also are of interest from the theoretical point of view as a tool to understand quantum field theory from a different perspective. However, boundary problems are also relevant in higher dimensional quantum field theories as descriptions of surface critical phenomena, and, as mentioned above, in the context of the Casimir effect.

One of the very interesting phenomena that can occur in the presence of a boundary is the existence of nontrivial ground state configurations, which approach the bulk vacuum only asymptotically, far away from the boundary. In a theory with a mass gap m in the bulk[2], any vacuum expectation value (VEV) of a field ϕ is bound to approach the bulk value ϕ_0 exponentially

$$\langle\phi(t,\underline{\mathrm{x}})\rangle_\alpha \sim \phi_0 + \bar{g}_\alpha\, \mathrm{e}^{-md} \tag{1.1}$$

as a function of the distance d from the boundary, where $\bar{g}_\alpha$ is a quantity characterising the boundary condition labelled by α. We mention that $\bar{g}$ itself is a physically relevant quantity in certain applications, e.g. in the context of thermal Coulomb plasma in contact with an ideal conductor electrode where it corresponds to the screened (or 'renormalized') surface charge density [2]. The existence of the $\bar{g}$ term in (1.1) can be related to the presence of a one-particle term in the boundary state of the form [3]

$$|B_\alpha\rangle = (1 + \tilde{g}_\alpha A^+(\underline{0}) + \dots)|0\rangle \tag{1.2}$$

where $A^+(\underline{\mathrm{k}})$ - with the normalization $[A(\underline{\mathrm{k}}),\, A^+(\underline{\mathrm{k}}')] = (2\pi)^D\sqrt{\underline{\mathrm{k}}^2+m^2}\,\delta(\underline{\mathrm{k}}-\underline{\mathrm{k}}')$ - denotes the operator that creates an asymptotic particle of momentum $\underline{\mathrm{k}}$, and $\tilde{g}_\alpha$ is just another quantity characterising the strength of the one-particle contribution to the boundary state. It is straightforward to derive (see subsection 3.2) the relation

$$\bar{g}_\alpha = \sqrt{\frac{Z}{2}}\,\tilde{g}_\alpha \tag{1.3}$$

where Z is the wavefunction renormalization constant of ϕ, considered as an interpolating field for the bulk asymptotic multi-particle states:

$$\langle 0|\phi(0)|A(\underline{\mathrm{k}}=\underline{0})\rangle = \sqrt{\frac{Z}{2}}$$

where $|0\rangle$ is the bulk vacuum state and $|A(\underline{\mathrm{k}}=\underline{0})\rangle$ is the asymptotic one-particle state containing a single particle with the lowest mass m and momentum zero. Note that Z is a bulk quantity which is independent of the boundary condition.

It was also shown in [4] that for $1+1$ dimensional QFTs the presence of a one-particle term in the boundary state leads to the following asymptotics of the ground state (Casimir) energy on a strip with boundary conditions α and β

$$E_{\alpha\beta}(L) \sim -m\tilde{g}_\alpha\tilde{g}_\beta \mathrm{e}^{-mL} \tag{1.4}$$

[2]In the following we always suppose that this is the case, i.e. we consider only theories with a non-vanishing mass gap, for which the formalism of asymptotic states is well defined.

while in the absence of such one-particle coupling the energy is expected to decay as e^{-2mL} (which was already known for the integrable case from studies of the thermodynamical Bethe Ansatz [5]). In subsection 3.5 we show that this is also valid in general $D+1$ spacetime dimensions, and provides a more convenient way of defining the quantity $\tilde{g}_\alpha$ than the expansion (1.2). The definition of $\tilde{g}_\alpha$ from the one-particle term in the boundary state depends on the convention of normalising the one-particle state, while the Casimir energy (1.4) is a directly measurable quantity independent of any conventions in the field theoretic formalism.

It was already noted in [1] that there is another manifestation of a one-particle coupling to the boundary, namely that the one-particle reflection factor has a pole at a special location. Introducing the rapidity parametrization of the $1+1$ dimensional energy-momentum $(e,\, p)$ as usual

$$e = m\cosh\theta \quad , \quad p = m\sinh\theta$$

it can be argued that the one-particle reflection factor off the boundary (which is the amplitude for the process involving a single particle both in the initial and in the final state) must have a pole at $\theta = i\frac{\pi}{2}$ with the residue denoted by

$$\operatorname*{Res}_{\theta=i\frac{\pi}{2}} R_\alpha(\theta) = i\frac{g_\alpha^2}{2} \tag{1.5}$$

which defines another characteristic quantity g_α (which is real for unitary theories)[3]. Ghoshal and Zamolodchikov identified g_α with $\tilde{g}_\alpha$, but later Dorey et al. [3], investigating one-point functions in the so-called scaling Lee-Yang model found numerically the relation

$$\tilde{g}_\alpha = \frac{g_\alpha}{2} \tag{1.6}$$

In the previous papers [4, 7, 8] we presented evidence (both analytic and numerical) that this relation extends to all $1+1$ dimensional integrable quantum field theories. Strictly speaking, however, there existed no field theoretic derivation or proof of this relation up to now. It was not clear either whether this relation can be extended to general boundary QFTs.

The knowledge of the relation between the g parameters has another interesting application. Namely, the classical vacuum expectation value and so the classical limit of $\bar{g}_\alpha$ can be calculated explicitely by solving the classical field equations. Relating this quantity to the residue of the reflection factor (g_α) makes it possible to identify the boundary condition corresponding to reflection factors found by solving the boundary bootstrap conditions, which is the fundamental idea underlying the work by Fateev and Onofri in [9].

The central aim of the present work is to extend these results already known in $1+1$ dimensional integrable QFTs, to nonintegrable theories and further to boundary QFTs defined in any spacetime dimensions. Even in the case of integrable theories, the derivation of (1.6) presented here is the first one that is truly general and starts from "first principles" (i.e. does not depend on additional assumptions or approximations).

We start in section 2 by reviewing the status in $1+1$ dimensional QFTs. We present arguments in the integrable case, further to those already made in [4], and then extend our considerations to the nonintegrable case using semiclassical techniques, where we show that both (1.3) and (1.6) can be extended to this case. However, there is no way to tackle general nonintegrable quantum field theories without further theoretical developments.

[3]Note that the above g quantities have nothing to do with the boundary entropy introduced in [6] which is also usually denoted by g and is often called the g-function.

In section 3, therefore, we develop the necessary tools to address the problem, using only general concepts of quantum field theory which are valid in any spacetime dimensions. In order to develop the boundary state formalism, in the appendix we derive a set of reduction formulae relating the matrix elements of the boundary state with bulk asymptotic many-particle states to correlation functions. Using them we can describe the boundary state in terms of (analytic continuation of) the boundary scattering amplitudes, analogously to what Ghoshal and Zamolodchikov did in the case of $1+1$ dimensional integrable field theories [1]. We present the detailed derivation of the one-particle term in the boundary state which leads us to the relation (1.3), and of the two-particle term, which is given in terms of the one-particle reflection factor. We then use a cluster argument to relate the singularity of the reflection factor to the one-point function, i.e. g to $\bar{g}$. For the case of $1+1$ dimensions this gives a proof of relation (1.6) "from first principles", valid for integrable and nonintegrable QFTs as well, while in the case of $D+1$ dimensional theories it allows us to characterize the nature and strength of the appropriate singularity of the reflection factor. In subsection 3.5 we give the derivation underlying the claim in our previous paper [10] where we gave a universal expression for the leading asymptotics of the Casimir energy (in the planar situation) valid also for nontrivially interacting theories, and also show how these results are affected by the presence of one-particle couplings to the boundary. We draw our final conclusions in section 4.

2 Examples

In this section we consider several examples as well as some heuristic derivations (under various assumptions) of the conjectured relations among the various g-s. Some of the examples and derivations are worked out in specific models, while some of the considerations are model independent. Part of the material in this section is on the level of (semi) classical considerations while the other is on the level of full QFT. All the examples are worked out in $1+1$ dimensions; they intended to provide a clear understanding of the underlying physics, and motivation for the derivations presented in the section 3, which are model independent and valid for generic number of spacetime dimensions.

2.1 Quantum theory examples

2.1.1 Boundary sine-Gordon model with Dirichlet boundary conditions

Our first example is the boundary sine-Gordon model with Dirichlet boundary conditions. In this model the bulk Lagrangian is written as

$$\mathcal{L}_{SG}(x,t)=\frac{1}{2}\left(\partial_\mu\Phi\right)^2-\frac{m^2}{\beta^2}\left(1-\cos\beta\Phi\right) \tag{2.1}$$

where β is the bulk coupling, m is the mass parameter and the field satisfies the boundary conditions

$$\Phi(x,t)|_{x=0}=\Phi_0^D\,,\qquad \Phi(x,t)|_{x=L}=\Phi_L^D$$

The fundamental range of the parameters $\Phi_{0,L}^D$ is given by $0\leq\frac{\beta}{2}\Phi_{0,L}^D\equiv\varphi_{0,L}\leq\frac{\pi}{2}$. In the regime $\beta^2<4\pi$ the lightest particle in the spectrum is the first breather, and it has the following reflection amplitude off the boundary [11]

$$R_{|\rangle}^{(1)}(\theta)=\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2\lambda}+1\right)}{\left(\frac{1}{2\lambda}+\frac{3}{2}\right)}\frac{\left(\frac{\eta}{\pi\lambda}-\frac{1}{2}\right)}{\left(\frac{\eta}{\pi\lambda}+\frac{1}{2}\right)}\,,\qquad (x)=\frac{\sinh\left(\frac{\theta}{2}+i\frac{\pi x}{2}\right)}{\sinh\left(\frac{\theta}{2}-i\frac{\pi x}{2}\right)} \tag{2.2}$$

Here θ is the rapidity of B^1 while

$$\lambda=\frac{8\pi}{\beta^2}-1\,,\qquad \text{and}\qquad \eta_{0,L}=\mp(\lambda+1)\varphi_{0,L}$$

are the parameters characterising the bootstrap solution [1]. Note that $R_{|\rangle}^{(1)}(\theta)$ has a first order pole at $\theta=i\frac{\pi}{2}$ originating from the $\left(\frac{1}{2}\right)$ factor. This results in the following coupling strength between the first breather and the boundary:

$$g_1\left(\eta\right)=2\sqrt{\frac{1+\cos\frac{\pi}{2\lambda}-\sin\frac{\pi}{2\lambda}}{1-\cos\frac{\pi}{2\lambda}+\sin\frac{\pi}{2\lambda}}}\tan\frac{\eta}{2\lambda} \tag{2.3}$$

(The expression under the square root is always positive as long as $\lambda>1$ which is necessary for the first breather to exist in the spectrum). However, some care must be taken, because the sign of the coupling g_1 must be opposite on the two ends of the strip since these are related by a spatial reflection under which the first breather is odd. As a result, eqn. (1.4) and eqn. (1.6) predict that the leading finite size correction to the ground state energy on the strip is given by

$$E(\varphi_0,\varphi_L)=m_1\frac{g_1(\eta_0)g_1(\eta_L)}{4}e^{-mL}+\dots \tag{2.4}$$

In [4] this expression was checked by comparing it to the exact ground state energy at large but finite L-s. The exact ground state energy of the boundary sine-Gordon model with Dirichlet boundary conditions was computed numerically (to high precision) from the NLIE proposed recently [12] and the detailed numerical comparison showed an excellent agreement. This comparison was later extended to more general boundary conditions in [13].

2.1.2 BTBA in the infrared limit

In this section we calculate the infrared limit of boundary thermodynamic Bethe Ansatz (BTBA) to support the relation (1.6). Although the original derivation of BTBA given by Leclair et al. [5] is only valid for the case when there is no one-particle term in the boundary state, it can be easily argued that the presence of one-particle terms makes no difference to the end result, and this is also supported by numerical studies using comparison with truncated conformal space (TCS) by Dorey et al. in [14].

In a theory with a single particle of mass m on a strip of length L the BTBA equation for the pseudo energy $\epsilon(\theta)$ takes the form

$$\epsilon\left(\theta\right)=2mL\cosh\theta-\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\Phi\left(\theta-\theta'\right)\log\left(1+\chi\left(\theta'\right)\mathrm{e}^{-\epsilon(\theta')}\right) \tag{2.5}$$

where $\Phi\left(\theta\right)=-i\frac{d}{d\theta}\log S\left(\theta\right)$ is the derivative of the phase of the two-particle S-matrix $S(\theta)$, and $\chi\left(\theta\right)=\bar{K}_\alpha\left(\theta\right)K_\beta\left(\theta\right)$ where

$$K_\alpha(\theta)=R_\alpha\left(i\frac{\pi}{2}-\theta\right)$$

and $\bar{K}_\alpha(\theta) = K_\alpha(-\theta)$. Once $\epsilon(\theta)$ is obtained, the energy of the ground state is given by

$$E_{\alpha\beta}(L) = -m\int_{-\infty}^{\infty}\frac{d\theta}{4\pi}\cosh\theta\log\left(1+\chi\left(\theta\right)\mathrm{e}^{-\epsilon(\theta)}\right)$$

In a theory with nonzero one-particle coupling on both boundaries χ has a second-order pole at $\theta = 0$. The logarithmic terms in (2.5) and in the ground state energy remain integrable and the BTBA makes perfectly good sense, but to obtain the correct asymptotic ($L \to \infty$) expression one needs to be careful. For large L we finally obtained in [4]

$$E_{\alpha\beta}(L) = -m\frac{|g_\alpha g_\beta|}{4}\mathrm{e}^{-mL} - m\int_{-\infty}^{\infty}\frac{d\theta}{4\pi}\cosh\theta\log\left(\frac{1+\chi\left(\theta\right)\mathrm{e}^{-2mL\cosh\theta}}{1+\frac{g_\alpha^2 g_\beta^2}{4\sinh^2\theta}\mathrm{e}^{-2mL}}\right) \tag{2.6}$$

The remaining integral is an expression of order e^{-2mL}, while the leading term agrees with (1.4) (for $\tilde{g}_\beta = g_\beta/2$) if $g_\alpha g_\beta > 0$. It was already noted in [14] (using a comparison with truncated conformal space method) that the BTBA equation only gives the correct ground state energy in this case.

Here we extend this result to values of boundary parameters such that $g_\alpha g_\beta < 0$, applying a suitable analytic continuation. Note that in the definition of the boundary coupling g_β a branch of the square root function must be chosen. In all known cases (e.g. Lee-Yang in [14] or sine-Gordon) there exists a branch choice such that the boundary couplings depend analytically on the boundary parameters. In this case a straightforward analytic continuation of the TBA equations (following the ideas originally proposed in [15]) from the domain where $g_\alpha g_\beta > 0$ to the one with $g_\alpha g_\beta < 0$ gives the ground state energy as

$$E_{\alpha\beta}(L) = m\sin u - m\int_{-\infty}^{\infty}\frac{d\theta}{4\pi}\cosh\theta\log\left(1+\chi\left(\theta\right)\mathrm{e}^{-\epsilon(\theta)}\right)$$

where now $\epsilon(\theta)$ is the solution of the equation

$$\epsilon\left(\theta\right) = 2mL\cosh\theta - \log\frac{S(\theta-iu)}{S(\theta+iu)} + \int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\Phi\left(\theta-\theta'\right)\log\left(1+\chi\left(\theta'\right)\mathrm{e}^{-\epsilon(\theta')}\right) \tag{2.7}$$

and u is determined by the equation

$$\bar{K}_\beta(iu)K_\alpha(iu)e^{-\epsilon(iu)} = -1$$

For L large one can put $\epsilon(\theta) = 2mL\cosh\theta$, yielding $\sin u \sim u \sim \frac{|g_\alpha g_\beta|}{2}e^{-mL}$, which in the energy expression just flips the sign of the term coming from the integral part. Thus

$$E_{\alpha\beta}(L) = -m\frac{g_\alpha g_\beta}{4}\mathrm{e}^{-mL} + \dots$$

in the parameter range where $g_\alpha g_\beta < 0$, which is consistent with eqns (1.6,1.4) which is the result of the cluster expansion [4].

2.1.3 Bethe-Yang equation

The aim of the following considerations is to emphasize the difference between the poles of the reflection amplitude corresponding to boundary bound states and the one appearing at $\theta = i\pi/2$, and to give an alternative argument for the validity of eqn. (1.6).

Suppose that in the half line theory there is a boundary bound state which is described by a pole of the reflection amplitude

$$R_\alpha(\theta) \sim \frac{i\Gamma^2/2}{\theta - iv_0}$$

in the physical strip $0 < v_0 < \frac{\pi}{2}$. The simplest way to determine the leading finite size correction to the energy of the bound state is to confine the theory to a strip of width L by imposing identical boundary conditions at the ends of the strip and look for purely imaginary solutions of the Bethe-Yang equation of the fundamental particle[4]

$$R_\alpha(\theta)R_\alpha(\theta)e^{i2mL\sinh\theta} = 1 \tag{2.8}$$

In order to describe the boundary bound states we continue this equation to imaginary values of the rapidity θ as in [17]. Writing $\theta = iv_0 + i\delta$ and assuming that $\delta \to 0$ for $L \to \infty$ one finds *two* solutions for δ such that the corresponding θ-s are in the physical strip, with the following large volume asymptotics:

$$\delta \sim \pm\frac{\Gamma^2}{2}e^{-mL\sin v_0}$$

The energy of these two solutions above the ground state is

$$E_\pm \sim m\cos v_0 \mp m\frac{\Gamma^2}{2}\sin v_0\, e^{-mL\sin v_0} \tag{2.9}$$

The interpretation of this result is clear: the two energies correspond to the two (approximate) wave functions containing the symmetric (resp. antisymmetric) combinations of the identical 'half line' bound states localized in the vicinity of the two boundaries (cf. [17]).

For solutions of the Bethe-Yang equation (2.8) that for $L \to \infty$ go to $\theta = i\frac{\pi}{2}$, the asymptotic behaviour is determined by the residue (1.5). Furthermore, writing $\theta = i\frac{\pi}{2} + i\delta$, only the solution with $\delta < 0$ is inside the physical strip. Therefore in this case we find the following result for the energy

$$E_{BY} = m\cosh\theta = m\frac{g_\alpha^2}{2}e^{-mL}$$

In contrast to the case of a boundary bound state discussed above, the other solution is nonphysical. Remembering that E_{BY} is the energy of the given state relative to the finite volume ground state of the system, it has a different interpretation than in the previous case. Since the true ground state is the symmetric combination of the asymptotic ground states localized at the left/right boundary, while the excited one is the antisymmetric one [7], E_{BY} can be identified with the *difference* between the energies of these two states, between which one can switch by changing the relative sign between the two $\tilde{g}$-s. Using (1.4) we obtain the relation

$$\Delta E = 2m\tilde{g}_\alpha^2 e^{-mL} \equiv E_{BY} = m\frac{g_\alpha^2}{2}e^{-mL}$$

Note that formally this result is just half of the energy difference $E_+ - E_-$ in (2.9) which is due to the special status of the $i\frac{\pi}{2}$ pole. This is again consistent with eqn. (1.6).

[4] This equation neglects vacuum polarization corrections, but one can argue that for large L-s it correctly gives the leading contribution - this was checked in [16] for the boundary sine-Gordon model.

2.1.4 Connecting the VEV and the boundary state

Next we consider the connection between the vacuum expectation value ${}_\beta\langle 0|\Phi(x,t)|0\rangle_\beta$ of a field Φ which is an interpolating field for the asymptotic particle states (where $|0\rangle_\beta$ denotes the ground state of the half line $x \leq 0$ theory with the boundary condition β imposed at $x=0$) and the coefficient(s) of the expansion of the boundary state, eqn. (1.2), in the crossed channel. Following the ideas presented in [3], this connection is based on the fundamental (defining) property of the boundary state, namely on the equivalence of the correlation functions in the two channels. Indeed this equivalence applied to the one-point function reads

$$ {}_\beta\langle 0|\Phi(x,t)|0\rangle_\beta = \langle 0|\Phi(y,\tau)|B_\beta\rangle \qquad y = it, \quad x = i\tau $$

Substituting eqn. (1.2) one finds

$$ {}_\beta\langle 0|\Phi(x,t)|0\rangle_\beta = \langle 0|\Phi(0)|0\rangle + \langle 0|\Phi(0)|\theta=0\rangle \tilde{g}_\beta e^{im\tau} + \dots $$

where the first term is the v.e.v. of the field in the bulk (which is an x independent constant) and the dots stand for the contribution of the multi-particle terms. The matrix element of the field in the second term is basically the normalization of the bulk one particle form factor in the closed channel

$$ \langle 0|\Phi(0)|\theta=0\rangle = \sqrt{\frac{Z}{2}} \tag{2.10} $$

with Z denoting the wavefunction renormalization constant of Φ. Exploiting the connection between the coordinates in the two channels one finds finally the asymptotic ($x \to -\infty$) expression

$$ {}_\beta\langle 0|\Phi(x,t)|0\rangle_\beta = \langle 0|\Phi(0)|0\rangle + \sqrt{\frac{Z}{2}}\tilde{g}_\beta e^{mx} + \mathcal{O}(e^{2mx}) \tag{2.11} $$

which gives the following relation

$$ \bar{g}_\beta = \sqrt{\frac{Z}{2}}\tilde{g}_\beta \tag{2.12} $$

2.2 (Semi)classical examples

Most of the examples and considerations presented so far relied on the integrability of the underlying model. In order to extend the relations between the various g-s to nonintegrable models we now present some further examples, and also develop methods which do not need integrability. The general framework in this section is that of the (semi)classical approximation of the various field theoretical models.

2.2.1 Connecting the classical VEV and the ground state energy

We consider a model described by a scalar field Φ on a strip of width L satisfying Dirichlet boundary conditions $\Phi(0,t) = \Phi_0$ and $\Phi(L,t) = \Phi_L$ with the bulk (Minkowski) action

$$ \mathcal{A}_{\text{bulk}} = \int dtdx \left\{ \frac{1}{2}(\partial_t\Phi)^2 - \frac{1}{2}(\partial_x\Phi)^2 - U(\Phi) \right\} \tag{2.13} $$

The ground state is a solution of the static classical equation of motion (satisfying also the boundary conditions)

$$ \frac{1}{2}\left(\frac{\partial\Phi}{\partial x}\right)^2 - U(\Phi) = C, \qquad U(\Phi) \geq 0 \tag{2.14} $$

where $U(\Phi)$ describes the self interaction of the scalar field and C is (an L dependent) constant of integration. For $L \to \infty$ this constant vanishes ($C \to 0$) and the ground state tends to a superposition of two 'half line' solutions determined by the two boundary conditions. This ground state has a simple qualitative description if we assume that $U(\Phi)$ has a minimum at Φ_* (normalized as $U(\Phi_*) = 0$) and that $\Phi_{0,L}$ are in the 'vicinity' of Φ_* (meaning that no other minimum is between Φ_* and Φ_0 or Φ_L): the scalar field starts at Φ_0 then increases (decreases) towards Φ_*, reaches a maximum (minimum) Φ_1 close to Φ_*, then increases/decreases to Φ_L[5]. It is important to notice that the L dependence of the ground state energy can be determined simply from that of the constant C. Indeed, writing

$$E(L) = \int_0^L dx \left\{ \frac{1}{2}\left(\frac{\partial \Phi}{\partial x}\right)^2 + U(\Phi) \right\} = -CL + \left\{ \int_{\Phi_0}^{\Phi_1} + \int_{\Phi_L}^{\Phi_1} \right\} dv \sqrt{2(U(v)+C)}$$

and exploiting

$$L = \left\{ \int_{\Phi_0}^{\Phi_1} + \int_{\Phi_L}^{\Phi_1} \right\} \frac{dv}{\sqrt{2(U(v)+C)}}$$

we get

$$\frac{dE}{dL} = -C, \qquad \text{thus} \qquad E(L) = E_\infty - \int C(L) dL \tag{2.15}$$

Therefore we determine next the large L asymptotics of the integration constant $C(L)$ using the qualitative picture of the ground state. If, in the half line theory, far away from the boundary, the solutions with the Φ_0 (Φ_L) b.c. have the asymptotic form

$$\Phi(x) \sim \Phi_* + \bar{g}_0 e^{-mx}, \qquad \Phi(x) \sim \Phi_* + \bar{g}_L e^{-mx} \tag{2.16}$$

then for large L-s, in the *central part* of the strip, which is far away from both boundaries, the scalar field is given by (upto subleading exponential corrections)

$$\Phi(x) \sim \Phi_* + \bar{g}_0 e^{-mx} + \bar{g}_L e^{m(x-L)}$$

(Note that the bulk vacuum expectation value Φ_* appearing in (2.16) is a parameter of the bulk theory; the dependence on the boundary conditions enters only via the coefficients of the exponential terms). Expanding the scalar potential in the vicinity of Φ_* as $U(\Phi) = \frac{m^2}{2}(\Phi - \Phi_*)^2 + \mathcal{O}((\Phi - \Phi_*)^3)$, and using eqn. (2.14) in the *central part* of the strip, one readily finds

$$C(L) = -2m^2 \bar{g}_0 \bar{g}_L e^{-mL}$$

Using this in eqn. (2.15) gives the asymptotic expression of the ground state energy as

$$E(L) = E_\infty - 2m\bar{g}_0 \bar{g}_L e^{-mL}$$

where E_∞ is the sum of the energies of the two half line solutions having the asymptotic behaviour in (2.16). This gives

$$\bar{g} = \frac{\tilde{g}}{\sqrt{2}}$$

which is consistent with eqn. (2.12) since for the normalization of the field given in (2.13) the classical value of the wave function renormalization constant is $Z = 1$.

[5]This description is valid if Φ_0 and Φ_L are both smaller or both greater than Φ_*; if Φ_* is between them, then $\Phi_1 = \Phi_*$ and the solution is monotonic.

2.2.2 Semiclassical limit of sine-Gordon model

In this subsection we consider the semiclassical limit of sine-Gordon model on a strip with Dirichlet boundary conditions and determine explicitely the three characteristic g-s in terms of the parameters of the model, thus verifying the conjectured relations among them.

The bulk Lagrangian is written in eqn. (2.1) and for simplicity we consider first the special case when the model is restricted to the negative half line ($x \leq 0$) and the field satisfies the boundary condition $\Phi(x,t)|_{x=0} = \Phi_0^D$ where the parameter Φ_0^D is in its fundamental range $0 \leq \frac{\beta}{2}\Phi_0^D \equiv \varphi_0 \leq \frac{\pi}{2}$. In this case the (semi)classical ground sate is given by a 'half' soliton standing at the location required by the boundary condition

$$\Phi_{\rm bg}\left(x,a^+\right) = \frac{4}{\beta}\arctan\left(e^{m(x-a^+)}\right) \quad \text{where} \quad e^{-ma^+} = \tan\frac{\varphi_0}{2}$$

Recalling the definition of $\bar{g}$

$$\Phi_{\rm bg} \sim \bar{g}e^{mx} \quad \text{for} \quad x \to -\infty$$

we obtain from the explicit solution

$$\bar{g} = \frac{4}{\beta}\tan\frac{\varphi_0}{2} \tag{2.17}$$

To obtain the semiclassical limit of g coming from the reflection amplitude one has to determine the appropriate solutions of the differential equation describing the linearized fluctuations in the standing soliton background. This was done in [7], [18], and using these wave functions one obtains the classical reflection amplitude

$$R(q) = \frac{m-iq}{m+iq}\,\frac{iq+m\cos\varphi_0}{iq-m\cos\varphi_0}\,, \qquad q = m\sinh\theta$$

Surprisingly, $R(q)$ has a second order pole at $\theta = i\frac{\pi}{2}$, i.e. for $\theta \sim i\frac{\pi}{2}$ it can be written as

$$R(q) \sim -\frac{4\tan^2\left(\frac{\varphi_0}{2}\right)}{(\theta - i\frac{\pi}{2})^2}$$

We can explain this second order pole in the following way. Since in the semiclassical limit the elementary field excitations become identical to the first breather we should compare $R(q)$ to the (semi)classical limit of the first breather's reflection amplitude (on the ground state $|\rangle$) (2.2), which has a first order pole at $\theta = i\frac{\pi}{2}$. However, as shown in [18], in the (semi)classical limit $\lambda \to \infty$ the η boundary parameter should also be scaled as

$$\eta = \eta_{\rm cl}(1+\lambda)$$

keeping $\eta_{\rm cl}$ finite, which is related to the boundary value of the field via

$$\eta_{\rm cl} = \varphi_0 = \frac{\beta\Phi_0^D}{2}$$

In addition $R^{(1)}_{|\rangle}(\theta)$ also has a pole at $\hat{\theta} = i\left(\frac{\pi}{2} - \frac{\pi}{2\lambda}\right)$ coming from the $\left(\frac{1}{2\lambda} + \frac{3}{2}\right)$ factor in the denominator, corresponding to the *bulk* process of two B^1's fusing into a B^2 of zero energy (rapidity $\theta = i\pi/2$), which is then absorbed by the boundary. Clearly for $\beta \to 0$ ($\lambda \to \infty$)

$\hat{\theta} \to i\frac{\pi}{2}$ and one obtains a second order pole. Therefore for sufficiently small β (sufficiently large λ) in the vicinity of $\theta \sim i\frac{\pi}{2}$ the quantum reflection amplitude behaves as

$$R^{(1)}_{|\rangle}(\theta) \sim \frac{A}{(\theta - i\frac{\pi}{2})(\theta - i(\frac{\pi}{2} - u_1))}\,, \qquad \lim_{\lambda\to\infty} u_1 = \lim_{\lambda\to\infty} \frac{\pi}{2\lambda} = 0$$

In this case the residue of $R^{(1)}_{|\rangle}(\theta)$ at $\theta = i\frac{\pi}{2}$ is

$$\operatorname*{Res}_{\theta=i\frac{\pi}{2}} R^{(1)}_{|\rangle}(\theta) = -i\frac{A}{u_1}$$

while in the (semi)classical limit one obtains

$$R(\theta) \sim \frac{A}{(\theta - i\frac{\pi}{2})^2}$$

Combining these expressions with the definition of g one finally finds

$$i\frac{g^2}{2} := \operatorname*{Res}_{\theta=i\frac{\pi}{2}} R^{(1)}_{|\rangle}(\theta) = i\frac{64}{\beta^2}\tan^2\frac{\varphi_0}{2}$$

i.e.

$$g = 4\sqrt{\frac{8}{\beta^2}}\tan\frac{\varphi_0}{2} \sim 4\sqrt{\frac{\lambda}{\pi}}\tan\frac{\eta}{2\lambda} \tag{2.18}$$

(where we 'removed' the semiclassical limit in writing the last approximate equality) which coincides with the semiclassical limit of (2.3) as was found also in [4].

In [4, 19] the classical ground state energy of the sine-Gordon model satisfying the Dirichlet boundary conditions on a strip of width L

$$\Phi(0,t) = \Phi_0^D, \qquad \Phi(L,t) = \Phi_L^D$$

(in the sector of zero topological charge) was found to be

$$E(\varphi_0, \varphi_L, L) \sim E_\infty - \frac{32m}{\beta^2}\tan\frac{\varphi_0}{2}\tan\frac{\varphi_L}{2}e^{-l}$$

in the asymptotic regime $l = mL \gg 1$. Here E_∞ is the sum of the energies of the two asymptotic static 'half' solitons representing the (asymptotic) ground state on the strip. Defining the classical counterpart of $\tilde{g}$ in (1.4) as

$$E(L) = E_\infty - m\tilde{g}_0\,\tilde{g}_L e^{-mL}$$

and comparing to the explicit expression gives

$$\tilde{g}_i = \frac{4\sqrt{2}}{\beta}\tan\frac{\varphi_i}{2}\,, \qquad i = 0,\ L \tag{2.19}$$

Combining eqn. (2.17,2.18) and (2.19) one obtains for the semiclassical limit of the various g-s:

$$\bar{g}_i = \frac{g_i}{2\sqrt{2}} = \frac{\tilde{g}_i}{\sqrt{2}}\,, \qquad i = 0,\ L \tag{2.20}$$

This is consistent with eqn. (1.5) and (1.6) and - since $Z \to 1$ in the (semi)classical limit - also with eqn. (2.11).

Similar results were obtained in [9] for the case of general affine Toda field theories (ATFT) where the authors use a different normalization for the fields and the particle states than we do. To facilitate the comparison, we give the relations for the case $a_1^{(1)}$ (sinh-Gordon) here. The action in that paper is normalized as

$$\mathcal{A} = \int d^2x \left\{ \frac{1}{8\pi} (\partial_\mu \phi)^2 + \mu \left(e^{\sqrt{2}b\phi} + e^{-\sqrt{2}b\phi} \right) \right\}$$

which in our notation means that the classical wave function renormalization constant is $Z = 4\pi$. As a result, from (2.10) we have

$$\langle 0|\phi(0)|\theta = 0\rangle = \sqrt{2\pi} + O(b^2)$$

while their one-particle state , $|\theta\rangle_{\mathrm{FO}}$ is normalized according to

$$\langle 0|\phi(0)|\theta = 0\rangle_{\mathrm{FO}} = \sqrt{\pi} + O(b^2)$$

which means that $|\theta = 0\rangle = \sqrt{2}|\theta = 0\rangle_{\mathrm{FO}}$. The residue of the reflection factor is parameterized in [9] as

$$R(\theta) \sim \frac{D(b)^2}{\theta - i\frac{\pi}{2}}$$

which means that $D(b)$ can be written in our notation as

$$D(b) = \frac{g}{\sqrt{2}}$$

The form of the one-particle term in the boundary state given in [9]

$$|B\rangle = \left(1 + D(b)A^+_{\mathrm{FO}}(0) + \dots\right)|0\rangle = \left(1 + \frac{D(b)}{\sqrt{2}}A^+(0) + \dots\right)|0\rangle \quad \Rightarrow \quad \tilde{g} = \frac{D(b)}{\sqrt{2}}$$

is then consistent with the relation (1.6)[6]. In addition, from the asymptotics of the classical vacuum solution found in [9] the VEV parameter $\bar{g}$ in the classical limit $b \to 0$ reads

$$\bar{g}_{\mathrm{classical}} = \frac{1}{b} \lim_{b\to 0} \sqrt{\pi} b D(b) = \sqrt{2\pi}\tilde{g}_{\mathrm{classical}}$$

which is then consistent with (1.3). The results of [9] therefore provide a generalization of the arguments of the present subsection to the case of any ATFT.

2.2.3 Semiclassical description of spontaneously broken Φ^4 theory

In this subsection we consider the spontaneously broken Φ^4 theory restricted by Dirichlet boundary conditions to a strip of width L. Using semiclassical considerations we express explicitely the (semiclassical limit of the) three characteristic g-s in terms of the parameters of the model, thus verifying explicitely the conjectured relations among them in this *nonintegrable* case.

[6] We mention, however, that the two-particle term in the cluster expansion of $|B\rangle$ is not properly normalized in [9], but it plays no role in the issue at hand.

The bulk Lagrangian of this model can be written as

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial\Phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\Phi}{\partial x}\right)^2 - \frac{\lambda}{4}\left(\Phi^2 - \frac{m^2}{\lambda}\right)^2 \qquad (2.21)$$

where $\lambda > 0$ and m is real. Introducing $z = \frac{mx}{\sqrt{2}}$ the well known kink (anti-kink) solution 'standing' at $x = a$ can be written as

$$\Phi(x) = \pm\frac{m}{\sqrt{\lambda}}\tanh(z - z_0)\,, \qquad z_0 = \frac{ma}{\sqrt{2}}$$

It is also well known that the mass of the elementary particle in the 'vacuum' sector is $\mu = m\sqrt{2}$.

First consider this model restricted to the half line $x < 0$ (or $x > 0$) imposing Dirichlet boundary condition at $x = 0$:

$$\Phi(0, t) = \Phi_0$$

which makes it possible to determine the explicit form of two of the three g-s. The essential observation is that – for $-\frac{m}{\sqrt{\lambda}} < \Phi_0 < \frac{m}{\sqrt{\lambda}}$ at least – the (semi)classical ground state of the model, $\Phi_{\rm bg}$, is given by a bulk kink/anti-kink standing at the position required by the boundary condition

$$\mp\frac{m}{\sqrt{\lambda}}\tanh(z_0) = \Phi_0$$

This enables us to determine $\bar{g}$ which is the parameter characterizing the field in the ground state: choosing, say, the solution that far away from the boundary at $x = 0$ tends to the $\frac{m}{\sqrt{\lambda}}$ bulk ground state, then $\bar{g}$ is defined as

$$\lim_{x\to-\infty}\Phi_{\rm bg} = \frac{m}{\sqrt{\lambda}} + \bar{g}_- e^{\mu x}, \qquad \lim_{x\to\infty}\Phi_{\rm bg} = \frac{m}{\sqrt{\lambda}} + \bar{g}_+ e^{-\mu x}$$

depending on whether we restrict the model to the $x < 0$ or to the $x > 0$ half line. Using the explicit kink/anti-kink solution one obtains readily

$$\bar{g}_- = -\frac{2m}{\sqrt{\lambda}}e^{-2z_0}, \qquad \bar{g}_+ = -\frac{2m}{\sqrt{\lambda}}e^{2z_0} \qquad (2.22)$$

To obtain g which characterizes the singularity of the reflection amplitude, we restrict the model to the $x < 0$ half line and determine the appropriate solutions of the differential equation describing the linearized fluctuations in the standing soliton (kink) background. This differential equation is the same as in the bulk case; the only remaining task one is to combine the left and right moving bulk wave solutions to satisfy the boundary condition at $x = 0$. In the bulk case the solutions in the continuum spectrum (which is relevant here) can be written [20]

$$\eta_q(z) = e^{iqz}(3\tanh^2(z - z_0) - 1 - q^2 - 3iq\tanh(z - z_0))$$

where q is a real parameter that determines the frequency of the fluctuation through $\omega^2 = m^2(2 + q^2/2)$. We look for solutions in the form

$$h_k(z) = \frac{\eta_k(z)}{2 - k^2 - 3ik}N_1 + \frac{\eta_{-k}(z)}{2 - k^2 + 3ik}N_2$$

that satisfy the boundary condition $h_k(0) = 0$ and also for which the coefficient of e^{ikz} far away from the boundary is one. The classical limit of the reflection amplitude is then defined by

$$\lim_{z \to -\infty} h_k(z) = e^{ikz} + R_{\rm cl}(k) e^{-ikz}$$

and from the explicit form of $h_k(z)$ one obtains:

$$R_{\rm cl}(k) = -\frac{2 - k^2 - 3ik}{2 - k^2 + 3ik} \frac{3\tanh^2(z - z_0) - 1 - k^2 + 3ik\tanh(z - z_0)}{3\tanh^2(z - z_0) - 1 - k^2 - 3ik\tanh(z - z_0)}\,, \qquad k = 2\sinh\theta$$

with θ being the usual rapidity parameter (in terms of which the spatial momentum of the particle is $p = \mu \sinh\theta$).

$R_{\rm cl}(k)$ has a second order pole at $\theta = i\frac{\pi}{2}$, i.e. for $\theta \sim i\frac{\pi}{2}$ it can be written as

$$R_{\rm cl}(k) \sim -\frac{12e^{-4z_0}}{(\theta - i\frac{\pi}{2})^2}$$

This second order pole can be explained in the same way as in the sine-Gordon case. We assume that in the full quantum reflection amplitude $R_{\rm q}(\theta)$ there is a pole close to $i\frac{\pi}{2}$ in such a way, that in the semiclassical limit (i.e. when $\lambda \to 0$) it coincides with the first order 'quantum' pole at $\theta = i\frac{\pi}{2}$:

$$R_{\rm q}(\theta) \sim \frac{A}{(\theta - i\frac{\pi}{2})(\theta - i(\frac{\pi}{2} - u))}\,, \qquad \lim_{\lambda \to 0} u = 0$$

Indeed in this case the residue of $R_{\rm q}(\theta)$ at $\theta = i\frac{\pi}{2}$ is

$$\operatorname*{Res}_{\theta = i\frac{\pi}{2}} R_{\rm q}(\theta) = -i\frac{A}{u} \tag{2.23}$$

while classically one obtains

$$R_{\rm cl}(k) \sim \frac{A}{(\theta - i\frac{\pi}{2})^2}$$

where

$$A = -12e^{-4z_0} \tag{2.24}$$

This mechanism can only work if in the *bulk* theory there is a fusion of appropriate bulk particles to generate the extra pole. Fortunately in the bulk spontaneously broken Φ^4 theory there is a rather complex semiclassical spectrum of 'approximate breathers' [20] and u can be identified with the fusion angle associated to the second lightest particle B_2 as a bound state of two copies of the lightest particle B_1:

$$2m_1 \cos u = m_2$$

Using the semiclassical formulae for m_1 and m_2 [20] we get

$$\cos u = \frac{m_2}{2m_1} \longrightarrow 1 - \frac{9}{32}\frac{\lambda^2}{m^4}$$

i.e.

$$u = \frac{3}{4}\frac{\lambda}{m^2} \tag{2.25}$$

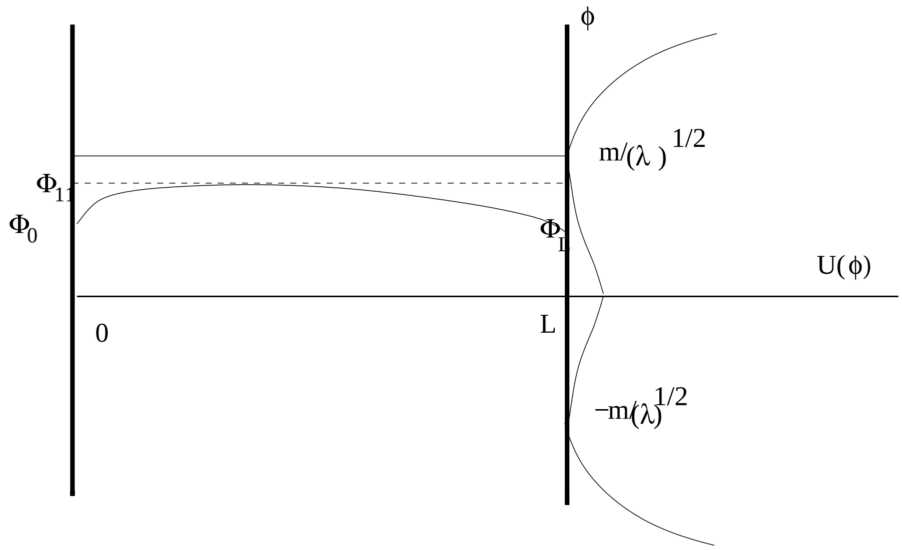

Figure 2.1: Qualitative behaviour of the classical ground state solution

for $\lambda \to 0$. Substituting (2.24,2.25) into the definition of g gives

$$i\frac{g^2}{2} = \underset{\theta=i\frac{\pi}{2}}{\mathrm{Res}}\, R_{\mathrm{q}}(\theta) = i\frac{16m^2}{\lambda}e^{-4z_0}, \qquad \text{i.e.} \qquad g = -m\frac{4\sqrt{2}}{\sqrt{\lambda}}e^{-2z_0} \tag{2.26}$$

To obtain the explicit form of the third g parameter we confine the model to a strip of width L by imposing Dirichlet boundary conditions ($\Phi(0,t) = \Phi_0$ and $\Phi(L,t) = \Phi_L$) and determine the classical ground state energy for asymptotically large L-s:

$$E(L) = E_\infty - \mu\tilde{g}_0\,\tilde{g}_L e^{-\mu L}$$

(In the $\lambda \to 0$ limit the classical energy gives the leading semiclassical contribution). Assuming $0 < \Phi_0\,, \Phi_L < \frac{m}{\sqrt{\lambda}}$ the qualitative behaviour of the static ground state solution is depicted in the figure 2.1 (a quantitative description can be given using the methods of [7]).

The ground state is a solution of the static classical equation of motion (2.14) with $U(\Phi) = \frac{\lambda}{4}(\Phi^2 - \frac{m^2}{\lambda})^2$. To obtain the L dependence of C explicitely we introduce $\Phi(x) = \frac{m}{\sqrt{\lambda}}\varphi(x)$, $\Phi_{0,1,L} = \frac{m}{\sqrt{\lambda}}\varphi_{0,1,L}$ and $C = -\frac{m^4}{4\lambda}d^2$, in terms of which the condition determining $d(L)$ is:

$$\frac{m}{\sqrt{2}}L = F(\phi(\varphi_0)|\alpha) + F(\phi(\varphi_L)|\alpha)\,, \qquad F(\phi(v)|\alpha) = \int_v^{\sqrt{1-d}} \frac{ds}{\sqrt{(1+d-s^2)(1-d-s^2)}} \tag{2.27}$$

where $F(\phi(v)|\alpha)$ is an elliptic integral of the first kind with modular angle α and amplitude $\phi(v)$ with

$$\sin\alpha = \sqrt{\frac{1-d}{1+d}}\,, \qquad \sin^2\phi = \frac{1+d}{1-d}\,\frac{1-d-v^2}{1+d+v^2}$$

For $L \to \infty$ one has $d \to 0$ and F has the asymptotic behaviour

$$F(\phi(v)|\alpha) = \frac{1}{2}\ln\frac{8}{d} - \ln\sqrt{\frac{1+v}{1-v}}$$

Using this in eqn. (2.27) determines the L dependence of d, which finally leads to

$$E(L)=E_{\infty}-8\mu\frac{m^2}{\lambda}\frac{1-\varphi_0}{1+\varphi_0}\,\frac{1-\varphi_L}{1+\varphi_L}e^{-\mu L}$$

The boundary conditions require $\tanh z_0=-\varphi_0$ and $\tanh z_L=\varphi_L$, leading to

$$\tilde{g}_0=-m\frac{2\sqrt{2}}{\sqrt{\lambda}}e^{2z_0}\,,\qquad \tilde{g}_L=-m\frac{2\sqrt{2}}{\sqrt{\lambda}}e^{-2z_L} \tag{2.28}$$

Combining eqn. (2.22,2.26) and (2.28) one obtains again that the semiclassical limits of the various g-s satisfy (2.20), i.e.

$$\bar{g}_i=\frac{g_i}{2\sqrt{2}}=\frac{\tilde{g}_i}{\sqrt{2}}\,,\qquad i=0,L$$

even in this nonintegrable model.

In passing we note that one can consider the Φ^4 theory in the symmetric phase (i.e. without spontaneous symmetry breaking, obtained formally by changing the sign of m^2 in (2.21)) along the same lines as we did in the previous subsections for the sine-Gordon and for the spontaneously broken Φ^4 theories here. In this case one can determine - using semiclassical considerations - the g coming from the VEV of the field in the ground state, as well as the g defined from the ground state energy on the strip and verify that they satisfy the conjectured relation. For the g determined from the reflection amplitude one encounters the following problem: although the differential equation of the small fluctuations around the ground state with the right boundary conditions can be solved and the classical reflection amplitude obtained, it exhibits the same second order pole at $\theta=i\frac{\pi}{2}$ as we encountered in the sine-Gordon and spontaneously broken Φ^4 theories. However, in the symmetric phase of Φ^4 theory there is no analogue of the bulk soliton/kink solutions, and thus neither of their breather bound states (exact or approximate). Despite this, the 'pole merging' mechanism described in the previous cases still exists, but the pole which collides with the one already located at $\theta=i\frac{\pi}{2}$ in the semiclassical limit comes from outside the physical strip, as appropriate analytic continuation of the reflection factors to this regime shows.

3 Boundary state formalism and proof of g-$\bar{g}$-$\tilde{g}$ relations in $D+1$ dimensions

In this section we analyze quantum field theories in $D+1$ dimensions in the presence of a D dimensional flat boundary. The correlators are defined via the path integral approach, and are then expressed using two alternative Hamiltonian descriptions of the system. If the boundary is in space ('open' channel) the information on the boundary condition is encoded in the reflection factors, which corresponds to the on-shell part of the correlators. However, when the boundary is situated in time ('closed' channel), the information is contained in the boundary state. By deriving a reduction formula in the 'closed' channel we are able to relate the matrix elements of this state to the on-shell part of the correlators and thus to the reflection factors. The knowledge of the boundary state, especially the one-particle boundary coupling, makes it possible to calculate the large distance behaviour of the one-point function, as well as to determine the leading finite size corrections of the ground state energy (Casimir effect).

Using the clustering property of the two-point function, these two quantities can be related to the singular part of the one-particle reflection amplitude. In this section we focus our attention on one- and two-point functions; the analysis of the multi-pont correlation functions is relegated to the appendix.

3.1 The concept of the boundary state

Consider an Euclidean quantum field theory of a scalar field Φ defined in a $D+1$ dimensional half spacetime, parameterized as $(x \leq 0, y, \vec{r})$, in the presence of a codimension one flat boundary located at $x = 0$. The correlation functions defined as

$$\langle \Phi(x_1, y_1, \vec{r}_1) \dots \Phi(x_N, y_N, \vec{r}_N) \rangle = \frac{\int \mathcal{D}\Phi \, \Phi(x_1, y_1, \vec{r}_1) \dots \Phi(x_N, y_N, \vec{r}_N) \, e^{-S[\Phi]}}{\int \mathcal{D}\Phi \, e^{-S[\Phi]}} \tag{3.1}$$

contain all information about the theory. The measure in the functional integral is provided by the classical action

$$S[\Phi] = \int d\vec{r} \int_{-\infty}^{\infty} dy \left[\int_{-\infty}^{0} dx \left(\frac{1}{2}(\partial_x \Phi)^2 + \frac{1}{2}(\partial_y \Phi)^2 + \frac{1}{2}(\vec{\partial}\Phi)^2 + U(\Phi) \right) + U_B(\Phi(x = 0, y, \vec{r})) \right]$$

which determines also the boundary condition via the boundary potential U_B.

This Euclidean quantum field theory can be considered as the imaginary time version of two different Minkowskian quantum field theories. We can consider $y = it$ as the imaginary time and so the boundary is located in space providing nontrivial boundary condition for the field Φ. We refer to this description as the *open channel*. The field Φ is an operator valued distribution that satisfies the equal time commutation relation

$$[\Phi(x, t, \vec{r}), \partial_t \Phi(x', t, \vec{r}\,')] = i\delta(x - x')\delta(\vec{r} - \vec{r}\,') \quad ; \qquad x, x' < 0$$

The space of states in this Hamiltonian description is the boundary Hilbert space $\mathcal{H}_B$ determined by the configurations on the equal time slices. $\mathcal{H}_B$ contains multi-particle states and is built over the boundary vacuum $|0\rangle_B$ by the successive application of the particle creation operators[7]. In the asymptotic past the particles do not interact and behave as free particles travelling towards the boundary; thus

$$\mathcal{H}_B = \left\{ a_{in}^{+}(k_1, \vec{k}_1) \dots a_{in}^{+}(k_N, \vec{k}_N)|0\rangle_B \quad , \qquad k_1 \geq \dots \geq k_N > 0 \right\}$$

where the operator $a_{in}^{+}(k, \vec{k})$, normalized as

$$[a_{in}(k, \vec{k}), \, a_{in}^{+}(k', \vec{k}\,')] = (2\pi)^D \omega(k, \vec{k})\delta(k - k')\delta(\vec{k} - \vec{k}\,') \quad ; \qquad k, k' > 0$$

creates an asymptotic particle of mass m with transverse momentum k and parallel momentum $\vec{k}$. The corresponding energy is $\omega(k, \vec{k}) = \sqrt{k^2 + \vec{k}^2 + m^2} = \sqrt{k^2 + m_{\text{eff}}(\vec{k})^2}$. In the Heisenberg picture the time evolution of the field

$$\Phi(x, t, \vec{r}) = e^{iH_B t}\Phi(x, 0, \vec{r})e^{-iH_B t}$$

[7]One can also introduce particle-like excitations confined to the boundary [21], but here we do not consider them for simplicity.

is generated by the following boundary Hamiltonian

$$H_B = \int d\vec{r} \left[\int_{-\infty}^{0} dx \left(\frac{1}{2}\Pi_t^2 + \frac{1}{2}(\partial_x \Phi)^2 + \frac{1}{2}(\vec{\partial}\Phi)^2 + U(\Phi) \right) + U_B(\Phi(x=0)) \right]$$

The correlator (3.1) can then be understood as the matrix element

$$\langle \Phi(x_1, y_1, \vec{r}_1) \dots \Phi(x_N, y_N, \vec{r}_N) \rangle = {}_B\langle 0| T_t \left(\Phi(x_1, t_1, \vec{r}_1) \dots \Phi(x_N, t_N, \vec{r}_N) \right) |0\rangle_B$$

where T_t denotes time ordering with respect to time t, and the vacuum $|0\rangle_B$ is normalized to 1.

Alternatively we can consider $x = i\tau$ as the Minkowskian time. In this case the boundary is located in time and we can use the usual infinite volume Hamiltonian description. This is referred to as the *closed channel*. The Hilbert space is the bulk Hilbert space $\mathcal{H}$ spanned by multi-particle *in* states

$$\mathcal{H} = \left\{ A_{in}^{+}(\kappa_1, \vec{k}_1) \dots A_{in}^{+}(\kappa_N, \vec{k}_N)|0\rangle \quad , \qquad k_1 \geq \dots \geq k_N \right\}$$

where the particle creation operators are normalized as

$$[A_{in}(\kappa, \vec{k}),\, A_{in}^{+}(\kappa', \vec{k}\,')] = (2\pi)^D \omega(\kappa, \vec{k}) \delta(\kappa - \kappa') \delta(\vec{k} - \vec{k}\,')$$

Time evolution

$$\Phi(\tau, y, \vec{r}) = e^{iH\tau} \Phi(0, y, \vec{r}) e^{-iH\tau}$$

is generated by the bulk Hamiltonian

$$H = \int d\vec{r} \int_{-\infty}^{\infty} dy \left(\frac{1}{2}\Pi_\tau^2 + \frac{1}{2}(\partial_y \Phi)^2 + \frac{1}{2}(\vec{\partial}\Phi)^2 + U(\Phi) \right)$$

The boundary appears in time as a final state in calculating the correlator (3.1):

$$\langle \Phi(x_1, y_1, \vec{r}_1) \dots \Phi(x_N, y_N, \vec{r}_N) \rangle = \langle B| T_\tau \left(\Phi(\tau_1, y_1, \vec{r}_1) \dots \Phi(\tau_N, y_N, \vec{r}_N) \right) |0\rangle$$

The state $\langle B|$ is called the boundary state, which is an element of the bulk Hilbert space and is defined by the equality of the two alternative Hamiltonian descriptions

$$\langle B| T_\tau \left(\Phi(\tau_1, y_1, \vec{r}_1) \dots \Phi(\tau_N, y_N, \vec{r}_N) \right) |0\rangle = {}_B\langle 0| T_t \left(\Phi(x_1, t_1, \vec{r}_1) \dots \Phi(x_N, t_N, \vec{r}_N) \right) |0\rangle_B$$

where the correspondence is valid if $(i\tau, y)$ is identified with (x, it). Using asymptotic completeness the boundary state can be expanded in the basis of asymptotic *in* states as

$$\langle B| = \langle 0| \left\{ 1 + \bar{K}^1 A_{in}(0,0) + \int_0^\infty \frac{d\kappa}{2\pi} \int \frac{d\vec{k}}{(2\pi)^{D-1}\omega(\kappa, \vec{k})} \bar{K}^2(\kappa, \vec{k}) A_{in}(-\kappa, -\vec{k}) A_{in}(\kappa, \vec{k}) + \dots \right\} \tag{3.2}$$

which we refer to as the cluster expansion for the boundary state (where due to translational invariance only bulk multi-particle states with zero total momentum can appear). The bars on top of the K coefficients indicate that the above expansion is that of the conjugate boundary state.

3.2 One-point function and K^1

The one-point function of the field, due to unbroken Poincaré symmetry along the boundary, only has a nontrivial dependence on x and can be written (using the 'open' channel formulation) as

$$_B\langle 0|\Phi(x,t,\vec{r})|0\rangle_B = G^1_{bdry}(x)$$

Going over to momentum space by Fourier transformation

$$_B\langle 0|\Phi(x,t,\vec{r})|0\rangle_B = \int \frac{d\omega}{2\pi}\int \frac{d\vec{k}}{(2\pi)^{D-1}}\int \frac{dk}{2\pi} e^{i(\omega t - kx - \vec{k}\vec{r})} G^1_{bdry}(\omega,k,\vec{k})$$

and analyzing the all order perturbative expression the following form is obtained:

$$G^1_{bdry}(\omega,k,\vec{k}) = (2\pi)^D\delta(\omega)\delta(\vec{k})G^2_{bulk}(\omega,k,\vec{k})\left[2\pi\delta(k)G^1_{bulk} + B^1_{bdry}(k)\right]$$

Unbroken translational invariance along the boundary is manifested in the conservation of energy and parallel momentum. Summing up the contributions of the outer leg the bulk two-point function, a pre-factor G^2_{bulk} can always be isolated. The remaining factor has a part that preserves the transverse momentum (which is the same as the bulk one-point function G^1_{bulk}) and another one which depends on the boundary condition and violates transverse momentum conservation. The bulk two-point function has the well-known Källen-Lehmann representation in terms of the spectral function $\sigma(m)$

$$G^2_{bulk}(\omega,k,\vec{k}) = \frac{iZ}{\omega^2 - k^2 - \vec{k}^2 - m^2 + i\epsilon} + \int_{2m}^{\infty} dm' \frac{i\sigma(m')}{\omega^2 - k^2 - \vec{k}^2 - m'^2 + i\epsilon}$$

where the wave function renormalization constant Z characterizes the strength of the on-shell part. Performing the energy-momentum integration and picking up the pole terms the one-point function can be expressed as

$$_B\langle 0|\Phi(x,t,\vec{r})|0\rangle_B = \langle 0|\Phi(0)|0\rangle - \frac{iZ}{2m}B^1_{bdry}(im)e^{mx} - \int_{2m}^{\infty} dm' \frac{i\sigma(m')}{2m'}B^1_{bdry}(im')e^{m'x} \quad (3.3)$$

From this we can read off the leading large distance behaviour (dominated by the on-shell part):

$$_B\langle 0|\Phi(x,t,\vec{r})|0\rangle_B = \langle 0|\Phi(0)|0\rangle + \bar{g}e^{mx} \quad ; \qquad \bar{g} = -\frac{iZ}{2m}B^1_{bdry}(im) \quad (3.4)$$

Our aim now is to connect the quantity $\bar{g}$ to the coefficient K^1 of the boundary state (3.2). This can be accomplished by calculating the matrix element

$$\langle B|A^+_{in}(k,\vec{k})\rangle = (2\pi)^D\omega(\kappa,\vec{k})\delta(\kappa)\delta(\vec{k})\bar{K}^1$$

in the closed channel. Using the reduction formula derived in the appendix, it can be expressed in terms of the one-point function as

$$\begin{aligned}\langle B|A^+_{in}(\kappa,\vec{k})\rangle \;=\; & \frac{i}{\sqrt{2Z}}\int_{-\infty}^{\infty} dy \int_{-\infty}^{0} d\tau \int d\vec{r}\, e^{-i\omega(\kappa,\vec{k})\tau + i\kappa y + i\vec{k}\vec{r}} \\ & \left\{\partial_\tau^2 - \partial_y^2 - \vec{\partial}^2 + m^2 - \delta(\tau)(\partial_\tau + i\omega(\kappa,\vec{k}))\right\}\langle B|\Phi(\tau,y,\vec{r})|0\rangle\end{aligned}$$

In the appendix it is also shown that only the on-shell part of the correlator contributes to the quantities K, and thus one can substitute

$$\langle B|\Phi(\tau,y,\vec{r})|0\rangle = {}_B\langle 0|\Phi(x=i\tau,t=-iy,\vec{r})|0\rangle_B \approx -\frac{iZ}{2m}e^{im\tau}B^1_{bdry}(im)$$

Plugging this expression back we obtain

$$\bar{K}^1 = \frac{-i}{m}\sqrt{\frac{Z}{2}}\left[G^1_{boundary}(im)\right] = \tilde{g}$$

from which using (3.4) we can establish the relation

$$\bar{g} = \sqrt{\frac{Z}{2}}\tilde{g}$$

We note that the derivation remains valid if the Lagrangian field Φ is replaced by any bulk interpolating field for the asymptotic particles and its appropriate wave function renormalization Z.

3.3 Two-point function and the relation between R^1_1 and K^2

The two-point function in the open channel (using the unbroken spacetime symmetries) can be written as

$$\begin{aligned}{}_B\langle 0|T(\Phi(x,t,\vec{r})\Phi(x^{'},t^{'},\vec{r}^{\,'}))|0\rangle_B \; &= \\ &\int\frac{d\omega}{2\pi}\int\frac{d\vec{k}}{(2\pi)^{D-1}}\int\frac{dk}{2\pi}\int\frac{dk^{'}}{2\pi}e^{i\omega(t-t^{'})-ikx-ik^{'}x^{'}-i\vec{k}(\vec{r}-\vec{r}^{\,'})}\tilde{G}^2_{bdry}(\omega,k,k^{'},\vec{k})\end{aligned}$$

after integrating out ω' and $\vec{k}'$ using the delta-functions in (A.3), where

$$\tilde{G}^2_{bdry}(\omega,k,k^{'},\vec{k}) = 2\pi\delta(k-k^{'})G^2_{bulk}(\omega,k,\vec{k}) + G^2_{bulk}(\omega,k,\vec{k})B^2_{bdry}(\omega,k,k^{'},\vec{k})G^2_{bulk}(\omega,k^{'},\vec{k})$$

Its on-shell part determines the two-particle reflection factor, which is the probability amplitude of a transition from a one-particle initial state $|k^{'},\vec{k}^{'}\rangle_{in} = a^+_{in}(k,\vec{k})|0\rangle_B$ into a one-particle final state $|k,\vec{k}\rangle_{out} = a^+_{out}(k,\vec{k})|0\rangle_B$ and can be expressed as the matrix element:

$${}_{out}\langle k,\vec{k}|k^{'},\vec{k}^{\,'}\rangle_{in} = (2\pi)^D\omega(k,\vec{k})\delta(k-k^{'})\delta(\vec{k}-\vec{k}^{\,'})R^1_1(\omega(k,\vec{k}),k)$$

where R^1_1 is the one-particle to one-particle reflection factor (cf. (A.5)). In order to simplify notation (and to conform better with the conventions of section 2) we drop the indices and denote R^1_1 simply by R from now on. The boundary reduction formula connects it to the correlator as[8]

$$\begin{aligned}{}_{out}\langle k,\vec{k}|k^{'},\vec{k}^{\,'}\rangle_{in} \; &= \; {}_{out}\langle k,\vec{k}|k^{'},\vec{k}^{\,'}\rangle_{out} \\ &-2Z^{-1}\int_{-\infty}^{0}dx\int dt\int d\vec{r}\,e^{-i\omega(k,\vec{k})t+i\vec{k}\vec{r}}\cos(kx)\left\{\partial_t^2-\partial_x^2-\vec{\partial}^2+m^2+\delta(x)\partial_x\right\} \\ &\int_{-\infty}^{0}dx\int dt\int d\vec{r}\,e^{-i\omega(k^{'},\vec{k}^{\,'})t^{'}+i\vec{k}^{\,'}\vec{r}^{\,'}}\cos(k^{'}x^{'})\left\{\partial_{t^{'}}^2-\partial_{x^{'}}^2-\vec{\partial}^{'2}+m^2+\delta(x^{'})\partial_{x^{'}}\right\} \\ &\qquad {}_B\langle 0|T_t(\Phi(x,t,\vec{r})\Phi(x^{'},t^{'},\vec{r}^{\,'}))|0\rangle_B\end{aligned}$$

[8]The normalization of the creation operators agrees with [10] but differs by a factor $\sqrt{2}$ compared to [21] and this affects the reduction formulae.

As was explained in [21] and is shown in the appendix only the on-shell part contributes. As a consequence we can keep the on-shell part from the bulk two-point function and represent the correlator equivalently as

$$
\begin{aligned}
{}_B\langle 0|T_t(\Phi(x,t,\vec{r})\Phi(x^{'},t^{'},\vec{r}^{\,'}))|0\rangle_B \quad &\approx \\
\int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{d\vec{k}}{(2\pi)^{D-1}}\frac{iZe^{-i\omega(t-t^{'})}e^{i\vec{k}(\vec{r}-\vec{r}^{\,'})}}{\omega^2-k^2-\vec{k}^2-m^2+i\epsilon}&\left[e^{-ik(x-x^{'})}+\frac{Z}{2k}B^2_{bdry}(\omega,k,k,\vec{k})e^{-ik(x+x^{'})}\right]
\end{aligned}
$$

Plugging this expression into the reduction formula, the bulk part of the two-point function cancels the disconnected piece and the reflection factor turns out to be

$$
R(\omega(k,\vec{k}),k)=\frac{Z}{2k}B^2_{bdry}(\omega(k,\vec{k}),k,k,\vec{k})
$$

Let us connect this quantity to the quantity K^2 in the boundary state (3.2), considering the following matrix element:

$$
\langle B|A^+_{in}(\kappa,\vec{k})A^+_{in}(\kappa^{'},\vec{k}^{\,'})\rangle=(2\pi)^D\omega(\kappa,\vec{k})\delta(\kappa+\kappa^{'})\delta(\vec{k}+\vec{k}^{\,'})\bar{K}^2(\kappa,\vec{k})
$$

where $\kappa>\kappa^{'}$ is assumed. Applying the result of the reduction formula presented in the appendix we have

$$
\begin{aligned}
\langle B|A^+_{in}(\kappa,\vec{k})a^+_{in}(A^{'},\vec{k}^{\,'})\rangle \quad &= \\
-(2Z)^{-1}\int_{-\infty}^{\infty}dy\int_{-\infty}^{0}d\tau\int d\vec{r}\,e^{-i\omega(\kappa,\vec{k})\tau+i\kappa y+i\vec{k}\vec{r}}&\left\{\partial_\tau^2-\partial_y^2-\vec{\partial}^{\,2}+m^2-\delta(\tau)(i\omega(\kappa,\vec{k})+\partial_\tau)\right\} \\
\int_{-\infty}^{\infty}dy^{'}\int_{-\infty}^{0}d\tau^{'}\int d\vec{r}^{\,\prime}\,e^{-i\omega(\kappa^{'},\vec{k}^{'})\tau^{'}+ik^{'}y^{'}+i\vec{k}^{\,'}\vec{r}^{\,'}}&\left\{\partial_{\tau^{'}}^2-\partial_{x^{'}}^2-\vec{\partial}^{\,'2}+m^2-\delta(\tau^{'})(i\omega(\kappa^{'},\vec{k}^{\,'})+\partial_{\tau^{'}})\right\} \\
&\langle B|T_\tau(\Phi(\tau,y,\vec{r})\Phi(\tau^{'},y^{'},\vec{r}^{\,'})|0\rangle
\end{aligned}
$$

Using the on-shell part of the two-point function and exchanging the role of space and time as $x=i\tau$ and $y=it$ we obtain

$$
\begin{aligned}
\langle B|T_\tau(\Phi(\tau,y,\vec{r})\Phi(\tau^{'},y^{'},\vec{r}^{\,\prime})|0\rangle \quad &= \quad {}_B\langle 0|T_t(\Phi(x,t,\vec{r})\Phi(x^{'},t^{'},\vec{r}^{\,\prime}))|0\rangle_B\approx \\
\int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{d\vec{k}}{(2\pi)^{D-1}}\frac{iZe^{ik(y-y^{'})}e^{i\vec{k}(\vec{r}-\vec{r}^{\,\prime})}}{\omega^2-k^2-\vec{k}^2-m^2+i\epsilon}&\left[e^{i\omega(\tau-\tau^{'})}+\frac{Z}{2i\omega}B^2_{bdry}(-ik,i\omega,i\omega,\vec{k})e^{i\omega(\tau+\tau^{'})}\right]
\end{aligned}
$$

where additionally we also exchanged the role of the energy and transverse momentum as $\omega\leftrightarrow -ik$. In the reduction formula the bulk part depending on $\tau-\tau^{'}$ does not contribute and we obtain

$$
\bar{K}^2(\kappa,\vec{k})=\frac{Z}{2i\omega(\kappa,\vec{k})}B^2_{bdry}(-i\kappa,i\omega(\kappa,\vec{k}),i\omega(\kappa,\vec{k}),\vec{k})
$$

which means that the relation between $\bar{K}^2$ and R is

$$
\bar{K}^2(\kappa,\vec{k})=R(\omega\to -i\kappa,k\to i\omega)
$$

In two spacetime dimensions one can use the rapidity parametrization

$$
\omega=m\cosh\vartheta \quad , \quad \kappa=m\sinh\vartheta
$$

and then

$$
\bar{K}^2(\vartheta)=R\left(i\frac{\pi}{2}+\vartheta\right)
$$

which is the same as the relation derived by Ghoshal and Zamolodchikov [1].

3.4 The connection between $\bar{g}$ and the reflection factor

The singularity property of the two-particle reflection factor generally follows from unitarity, similarly to the well-known bulk theory of the analytic S matrix. Unitarity of the reflection operator $\mathcal{R}$ can be expressed in terms of the interaction matrix $\mathcal{T}$ as

$$\mathcal{R}\mathcal{R}^{+}=(1+i\mathcal{T})(1-i\mathcal{T}^{+})=1$$

Calculating the one-particle matrix element of this relation between states $|k,\vec{k}\rangle_B$ and $|k^{'},\vec{k}^{'}\rangle_B$ and inserting a resolution of the identity we have

$$\begin{aligned}2\Im m\mathcal{T}(k,\vec{k})(2\pi)^D\omega(k,\vec{k})\delta(k-k^{'})\delta(\vec{k}-\vec{k}^{''}) \;&=\; {}_B\langle k,\vec{k}|\mathcal{T}|0\rangle_B\,{}_B\langle 0|\mathcal{T}^{+}|k^{'},\vec{k}^{'}\rangle_B+\\ \sum_n\int\frac{d\vec{q}}{(2\pi)^{D-1}\omega(iu_n(\vec{q}),\vec{q})}&{}_B\langle k,\vec{k}|\mathcal{T}|iu_n(\vec{q}),\vec{q}\rangle_B\,{}_B\langle iu_n(\vec{q}),\vec{q}|\mathcal{T}^{+}|k^{'},\vec{k}^{'}\rangle_B+\\ \int\frac{d\vec{q}}{(2\pi)^{D-1}}\int_0^{\infty}\frac{dq}{2\pi\omega(q,\vec{q})}&{}_B\langle k,\vec{k}|\mathcal{T}|q,\vec{q}\rangle_B\,{}_B\langle q,\vec{q}|\mathcal{T}^{+}|k^{'},\vec{k}^{'}\rangle_B+\dots\end{aligned}$$

where the second term on the right hand side corresponds to boundary bound states (degrees of freedom propagating along the boundary) which are pole singularities located at $k=iu_n\left(\vec{k}\right)$ with $0<u_n(\vec{k})<m$. Unitarity (as expressed above) gives the following relation for the residue of the reflection factor:

$$2\Im m\left[\omega(iu,\vec{k})R(iu,0)\right] \;=\; 2\pi\frac{\omega(iu,0)}{iu}\delta(k-iu)C_n\left(\vec{k}\right)C_n\left(\vec{k}\right)^{\dagger}$$

where $C_n\left(\vec{k}\right)$ is the strength of the on-shell vertex corresponding to the creation of the boundary bound state by the bulk particle.

Using the unbroken spacetime symmetries we can extract energy and parallel momentum conservation which in the first term gives

$${}_B\langle k,\vec{k}|\mathcal{T}|0\rangle_B={}_B\langle k,\vec{k}|\mathcal{R}|0\rangle_B=(2\pi)^D\delta(\vec{k})\delta(\omega(k,\vec{k}))R^1(im)$$

This term vanishes at physical energies but we can try to make an analytical continuation via the boundary reduction formula and express R^1 from the one-point function. The energy delta function can be rewritten as

$$\delta(\omega(k,\vec{0}))=\frac{\omega(k,\vec{0})}{k}\delta(k-im)$$

and the kinematical pre-factor vanishes exactly at the momentum value it is concentrated on. This shows that the singularity of the two-particle reflection matrix at $k=im$ and $\vec{k}=0$ is not a pole in the momentum variable k. We see that in this case the unitarity argument does not connect the singular part of the reflection factor to the one-particle emission amplitude and so to the on-shell part of the one-point function. The fact that there is an essential difference between the behaviour of this particular type of singularity and the ones associated to boundary bound states is consistent with their different treatment in the Bethe-Yang analysis presented in subsection 2.1.3. Therefore we choose an alternative way to connect the strength of the singularity in the reflection factor to the VEV of the field, making use of the clustering property of the two-point function.

In the Euclidean regime the on-shell part of the two-point function, which determines its behaviour in the asymptotic regime far from the boundary, reads as

$$
\begin{aligned}
&{}_B\langle 0|\,\Phi(x,y,\vec{r})\Phi(x',y',\vec{r}')\,|0\rangle_B \;\approx \qquad\qquad (3.5)\\
&\int \frac{d\rho}{2\pi}\frac{dk}{2\pi}\frac{d\vec{k}}{(2\pi)^{D-1}}\frac{Z}{\rho^2+k^2+\vec{k}^2+m^2}\mathrm{e}^{-i\rho(y-y')}e^{i\vec{k}(\vec{r}-\vec{r}')}\left\{\mathrm{e}^{-ik(x-x')}+R(k,\vec{k})\mathrm{e}^{-ik(x+x')}\right\}
\end{aligned}
$$

where we supposed $t>t'$ and introduced $\omega=-i\rho$, $t=iy$. Cluster property implies that for large temporal separation the Euclidean two-point function satisfies

$$
{}_B\langle 0|\,\Phi(x,y,\vec{r})\Phi(x',y',\vec{r}')\,|0\rangle_B = {}_B\langle 0|\,\Phi(x,y,\vec{r})\,|0\rangle_B\;{}_B\langle 0|\,\Phi(x',y',\vec{r}')\,|0\rangle_B + \mathrm{O}\left(\mathrm{e}^{-\mu|y-y'|}\right)
$$

with some characteristic scale μ that corresponds to the gap in the spectrum above the vacuum. The presence of the disconnected piece signals the nontrivial vacuum expectation value of the field Φ (which is time independent $\partial_y\langle 0|\,\Phi(x,y,\vec{r})\,|0\rangle=0$ due to y-translational invariance, and is also independent of $\vec{r}$ for a similar reason). In the asymptotic regime we then expect the following behaviour

$$
{}_B\langle 0|\,\Phi(x,y,\vec{r})\Phi(x',y',\vec{r}')\,|0\rangle_B \sim \tilde{g}^2\mathrm{e}^{m(x+x')} \qquad |y-y'|\to\infty \tag{3.6}
$$

In a free theory, with $Z=1$ and $R(k,\vec{k})=\pm1$ (corresponding to Neumann/Dirichlet boundary conditions), the integral (3.5) can be evaluated explicitely with the result

$$
\frac{1}{2\pi}\left(K_0\left(mr_-\right)\pm K_0\left(mr_+\right)\right)\;,\quad r_\pm=\sqrt{(y-y')^2+(x\pm x')^2+(\vec{r}-\vec{r}')^2}
$$

which decays exponentially when $|y-y'|\to\infty$ and so there is no disconnected piece. In fact, this remains true as long as $R(k,\vec{k})$ is regular at $k=\pm im$ as will be implied by the analysis below.

Here we have to make an Ansatz for the singularity type of the reflection factor. The expectation from its structure is that it provides the needed clustering, moreover it should match with the exact results available in two dimensions. In two dimensional integrable theories the singularity of the reflection factor at $\vartheta=\frac{i\pi}{2}$ is a pole in the rapidity variable ($k=m\sinh\theta$) of the form

$$
R(\vartheta)\sim\frac{ig^2/2}{\vartheta-i\frac{\pi}{2}}\sim-\frac{g^2/2}{\cosh\vartheta}
$$

Changing the rapidity variable to the momentum this corresponds to the behaviour

$$
R(k)\sim-\frac{mg^2/2}{\sqrt{k^2+m^2}}
$$

Observe that the singularity in the variable k is not of a pole type, but is milder, just as expected from the unitarity argument. In higher dimensions we expect this singularity to be connected with the virtual one-particle emission (just as in the 1+1 dimensional case), and due to parallel momentum conservation it is expected to have the following form:

$$
R(k,\vec{k})\sim-\frac{mg^2/2}{\sqrt{k^2+\vec{k}^2+m^2}}(2\pi)^D\delta(\vec{k}) \tag{3.7}
$$

Such a term introduces a singularity at $k^2 = -m^2$ where the on-shell value of ρ is 0 which means a contribution which is constant in the temporal separation $|y - y^{'}|$. We remark that in order to obtain this behaviour from the analytic R-matrix theory the classification and analysis of Coleman-Thun diagrams at this particular kinematical point is necessary which is outside the scope of the present considerations (for a general exposition see [21]).

We can calculate the effect of this singularity in the following way. For definiteness, let us specify $y > y^{'}$: then the integration contour in (3.5) can be closed in the $\Im m\, \rho < 0$ half-plane giving the result

$$_B \langle 0|\, \Phi(x,y,\vec{r})\Phi(x^{'},y^{'},\vec{r}^{'})\, |0\rangle_B = -\int \frac{dk}{2\pi} \frac{Z}{2\sqrt{k^2+m^2}} \mathrm{e}^{-\sqrt{k^2+m^2}(y-y')} \left\{ -\frac{mg^2/2}{\sqrt{k^2+m^2}} \mathrm{e}^{-ik(x+x')} \right\}$$

where we have also performed the trivial $\vec{k}$ integration. Remembering now that $x + x' < 0$ we can close the contour in this term in the lower half-plane which gives

$$_B \langle 0|\, \Phi(x,y,\vec{r})\Phi(x^{'},y^{'},\vec{r}^{'})\, |0\rangle_B \sim \frac{g^2}{8} Z \mathrm{e}^{m(x+x')} + \dots$$

and comparing to (3.6) it follows that

$$\bar{g} = \frac{g}{2}\sqrt{\frac{Z}{2}}$$

and so

$$\tilde{g} = \frac{g}{2}$$

It is useful to stress that for $D > 1$ this result shows just the consistency of the assumption (3.7). In $1+1$ dimensions, however, the exact reflection factors of many integrable theories are known explicitely, and exhibit a pole at $\vartheta = i\pi/2$, and in this case the cluster argument is in fact a field theoretic proof of a relation between g and $\tilde{g}$ (valid also for the nonintegrable case), conjectured in [3] and checked using both theoretical and numerical arguments in [4].

3.5 Finite size energy correction from the boundary state

We now calculate the ground-state energy per transverse volume, $E_0^{\alpha\beta}(L)$, of the system confined by two hyperplanes to the interval $0 \leq x \leq L$ and subject to boundary conditions labelled by α and β at the two ends. In doing so we compactify the other directions to circles of perimeter R with periodic boundary conditions and calculate the partition function in two different ways, corresponding to the two different Hamiltonian descriptions of the system introduced previously. In the open channel (where $y = -it$ is the imaginary time) the partition function can be written as

$$Z(L,R) = Tr(e^{-RH_{\alpha\beta}(L,R)})$$

which for large R behaves as

$$\lim_{R\to\infty} Z(L,R) = e^{-E_0^{\alpha\beta}(L)V} + \text{small correction}$$

where $V = R^D$. In the closed channel (where $x = it$ is the Euclidean time) the partition function is given by the following matrix element.

$$Z(L,R) = \langle B_\alpha | e^{-LH(R)} | B_\beta \rangle$$

Inserting a complete set of eigenstates of the periodic (bulk) Hamiltonian $H(R)$

$$Z(L,R)=\sum_n \frac{\langle B_\alpha|n\rangle\langle n|B_\beta\rangle}{\langle n|n\rangle}e^{-E_n(R)L}$$

We concentrate on the leading finite size correction to the ground state energy $E_0^{\alpha\beta}(L)$ and take L to be large. As a consequence the low lying energy levels dominate the sum:

$$Z(L,R)=e^{-E_0(R)L}\left[1+\sum_{\kappa,\vec{k}}\frac{\langle B_\alpha|\kappa,\vec{k}\rangle\langle\kappa,\vec{k}|B_\beta\rangle}{\langle\kappa,\vec{k}|\kappa,\vec{k}\rangle}e^{-\omega(\kappa,\vec{k})L}+\dots\right]$$

where the sum is over one-particle states with momentum $(\kappa,\vec{k})$. The finite volume restricts the momentum to be $\kappa=\frac{2\pi}{R}n$, and $k_i=\frac{2\pi}{R}n_i$ and the normalization of the creation operators becomes

$$[A_{in}(\kappa,\vec{k}),A_{in}^{+}(\kappa^{'},\vec{k}^{'})]=V\omega(\kappa,\vec{k})\delta_{\kappa,\kappa'}\delta_{\vec{k},\vec{k}'}$$

Using the form of the boundary state (3.2) yields

$$Z(L,R)=e^{-E_0(R)L}\left[1+mV\bar{K}_\alpha^1K_\beta^1e^{-mL}+\dots\right]$$

We normalize the ground-state energy with periodic boundary condition to zero: $E_0(R)=0$. As a result *the ground state energy per transverse volume* at leading order in L has the form

$$E_0^{\alpha\beta}(L)=-m\bar{K}_\alpha^1K_\beta^1e^{-mL}+\dots \tag{3.8}$$

If one of the K^1-s is zero then the leading correction comes from two-particle states. Although we explained this situation in [4, 10], for completeness we recall the derivation of the leading finite size correction. Using the cluster expansion (3.2), the partition function can be written as

$$Z(L,R)=e^{-E_0(R)L}\left[1+\sum_{\kappa,\vec{k},\kappa',\vec{k}'}\frac{\langle B_\alpha|\kappa,\vec{k},\kappa^{'},\vec{k}^{'}\rangle\langle\kappa,\vec{k},\kappa^{'},\vec{k}^{'}|B_\beta\rangle}{\langle\kappa,\vec{k},\kappa',\vec{k}'|\kappa,\vec{k},\kappa',\vec{k}'\rangle}e^{-\omega(\kappa,\vec{k})L-\omega(\kappa^{'},\vec{k}^{'})L}+\dots\right]$$

The spectrum of the possible $\kappa,\kappa^{'}$ has to be determined by solving the scattering problem of the two particles in volume V exactly. However, in the infinite volume limit the interaction between the particles can be neglected. (For large L the main contribution to the ground state energy comes from the low lying energy levels, for which this approximation becomes exact as $R\to\infty$.) Using the explicit form of the boundary state, the result for the Casimir energy (per unit transverse area) is

$$E_0^{\alpha\beta}(L)=-\int\frac{d\vec{k}}{(2\pi)^{D-1}}\int_0^\infty\frac{d\kappa}{2\pi\omega(\kappa,\vec{k})}\bar{K}_\alpha^2(\kappa,\vec{k})K_\beta^2(\kappa,\vec{k})e^{-2\omega(\kappa,\vec{k})L}$$

as we explained in our previous paper [10], where it was derived using the assumption that the modes indexed by $\vec{k}$ can be treated as independent two-dimensional degrees of freedom (their interaction only entering higher order terms explicitly). The present derivation, however, completely dispenses with this additional assumption.

4 Conclusion

In this paper we have considered the relation between one-point function, finite size corrections to ground state energy and the analytic structure of scattering amplitudes in a general boundary QFT. Our main result is the derivation "from first principles" of the relations (1.3,1.6) between the parameters characterizing these quantities for a general $D+1$ dimensional quantum field theory.

Besides giving a solid theoretical background to preexisting results, we developed the formalism of the boundary state for the $D+1$ dimensional case, together with a new set of reduction formulae relating the boundary state to the correlation functions, which can be used to express the cluster expansion of the boundary state in terms of the boundary scattering amplitudes.

The resulting formalism can be used to address various issues in boundary quantum field theory. In addition to the derivation of the relation between the g parameters and its extension to general field theories, we have shown that it can be used to derive the large volume asymptotics of the Casimir effect. In particular in the case when there is no one-particle coupling to the boundary, we proved that the result of [10] for the leading behaviour is indeed universal, even for interacting fields. We extended this result by giving the leading term in the Casimir energy (3.8) for the case with nonvanishing one-particle coupling, and used our results to relate it to the asymptotic behaviour of the vacuum expectation value of the interpolating field. It is important to note that it relates the planar Casimir effect to physical quantities (vacuum expectation values, reflection factor) that are calculable in the infinite volume boundary quantum field theory, which is a much simpler setting than the finite volume case. In addition, the expression is free of any ultraviolet divergences right from the start, the underlying reason being that it involves only relations between physical (i.e. renormalized) entities, as was already discussed in [10]. It also noteworthy that it gives a general formula for the dependence of the Casimir effect on the material properties of the boundary (characterized by the reflection factors) in the planar setting.

In principle, the cluster expansion for the boundary state (3.2) provides a systematic large volume expansion of the Casimir energy (and other quantities, such as vacuum expectation values) if higher particle terms are included. However, in the case when the reflection factor has a singularity corresponding to the one-particle coupling, even the two-particle term is divergent in general. This is a sort of infrared divergence, which in the case of the ground state energy is known to be eliminated by resummation of the series using e.g. thermodynamic Bethe Ansatz (cf. [4] where a regular large volume expansion is derived from boundary TBA), but this method only works for integrable theories (or for theories in $D+1$ dimensions with trivial bulk and completely elastic boundary scattering). It is plausible that something similar happens in the case of nonintegrable theories as well, but there is as yet no method to perform a resummation of the relevant terms, which remains an interesting open problem.

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A Reduction formulae

In this appendix we recall the derivation of the reduction formula in the open channel and show its physical meaning, namely that it connects the reflection factors to the on-shell part of the boundary Green functions. Then we derive an analogous reduction formula in the closed channel, and show that only the on-shell part of the boundary Green function contributes to the boundary state, thus a relation between the boundary state and the reflection factor is established.

A.1 Reduction formula in the open channel

The multi-particle reflection matrix is defined as the matrix element

$$\begin{aligned} R_k^l &= {}_B^{out}\langle p_1,\vec{p}_1;\dots;p_k,\vec{p}_k|q_1,\vec{q}_1;\dots;q_l,\vec{q}_l\rangle_B^{in} \\ &= {}_B^{out}\langle p_1,\vec{p}_1;\dots;p_k,\vec{p}_k|a_{in}^+(q_1,\vec{q}_1)|q_2,\vec{q}_2;\dots;q_l,\vec{q}_l\rangle_B^{in} \end{aligned} \tag{A.1}$$

The creation operator of the initial particle state can be written in terms of the asymptotic field as

$$a_{in}^+(k,\vec{k}) = -\frac{i}{\sqrt{2}}\int_{-\infty}^0 dx\cos(kx)\int d\vec{r}\, e^{i\vec{k}\vec{r}}\, e^{-i\omega(k,\vec{k})t}\overleftrightarrow{\partial_t}\,\Phi_{in}(x,t,\vec{r})$$

Using the property that the interpolating (interacting) field approaches the free asymptotic fields as

$$\Phi(x,t,\vec{r}) \to Z^{1/2}\Phi_{in/out}(x,t,\vec{r}) \quad \text{for} \quad t\to\mp\infty$$

we obtain the following form of the reflection matrix

$$\begin{aligned} R_k^l \;=\; & {}_B^{out}\langle p_1,\vec{p}_1;\dots;p_k,\vec{p}_k|a_{out}^+(q_1,\vec{q}_1)|q_2,\vec{q}_2;\dots;q_l,\vec{q}_l\rangle_B^{in} + \\ & i\sqrt{\frac{2}{Z}}\int_{-\infty}^0 dx\int d\vec{r}e^{i\vec{q}_1\vec{r}}\int dt\partial_t\{\cos(q_1x)e^{-i\omega(q_1,\vec{q}_1)t}\overleftrightarrow{\partial_t}\;{}_B\langle out|\Phi(x,t,\vec{r})|in\rangle_B\} \end{aligned}$$

where ${}_B\langle out|$, $(|in\rangle_B)$ is the shorthand form for ${}_B^{out}\langle p_1,\vec{p}_1;\dots;p_k,\vec{p}_k|$ and $|q_2,\vec{q}_2;\dots;q_l,\vec{q}_l\rangle_B^{in}$, respectively. It is necessary to be careful when performing the partial integration and keep the surface term. The connected part turns out to be

$$i\sqrt{\frac{2}{Z}}\int_{-\infty}^0 dx\int dtd\vec{r}e^{-i\omega(q_1,\vec{q}_1)t}e^{i\vec{q}_1\vec{r}}\cos(q_1x)\{\partial_t^2-\partial_x^2-\vec{\partial}^2+m^2+\delta(x)\partial_x\}\;{}_B\langle out|\Phi(x,t,\vec{r})|in\rangle_B \tag{A.2}$$

Repeating the same procedure one can express the reflection factor as the product of integro-differential operators acting on the Green functions. The connected part is related to the N-pont function, while the disconnected one appears when at least one of the incoming momentum (say $(q_i,\vec{q}_i)$) coincides with one of the outgoing momentum (say $(p_j,\vec{p}_j)$). It contains a delta function singularity $(2\pi)^D\omega(q_i,\vec{q}_i)\delta(q_i-p_j)\delta(\vec{q}_i-\vec{p}_j)$ whose coefficient is related to the $N-2$ point function. Here we would like concentrate on the connected part and to show, that the operator in (A.2) truncates the leg of the momentum space Green function and puts its momentum on-shell.

The momentum space Green function is defined by Fourier transformation

$$\begin{aligned} {}_B\langle 0|T_t\left(\Phi(x_1,t_1,\vec{r}_1)\dots\Phi(x_N,t_N,\vec{r}_N)\right)|0\rangle_B \;=\; & \\ \prod_{j=1}^N\int_{-\infty}^{\infty}\frac{d\omega_j}{2\pi}e^{i\omega_jt_j}\int_{-\infty}^{\infty}\frac{dk_j}{2\pi}e^{-ik_jx_j}\int_{-\infty}^{\infty}\frac{d\vec{k}_j}{(2\pi)^{D-1}}e^{-i\vec{k}_j\vec{r}_j}G_{bdry}^N(\{\omega_i,k_i,\vec{k}_i\}) & \end{aligned}$$

Inspecting the perturbative expansion we can always write

$$G^N_{bdry}(\{\omega_i, k_i, \vec{k}_i\}) = (2\pi)^D \delta(\sum_j \omega_j)\delta(\sum_j \vec{k}_j) \left[\text{disc.} + \prod_j G^2_{bulk}(\omega_j, k_j, \vec{k}_j) B^N_{bdry}(\{\omega_i, k_i, \vec{k}_i\}) \right] \tag{A.3}$$

where the disconnected part contains at least one particle which is unaffected by the boundary, that is a term proportional to $(2\pi)^D\delta(k_i - k_j)\delta(\vec{k}_i - \vec{k}_j)$. [9] Using the Källen-Lehman form of the bulk two-point function the contribution of a single leg can be written as

$$\int \frac{d\omega}{2\pi}\frac{dk}{2\pi}\frac{d\vec{k}}{(2\pi)^{D-1}} e^{i\omega t} e^{-ikx} e^{-i\vec{k}\vec{r}} \tag{A.4}$$

$$\left[\frac{iZ}{\omega^2 - k^2 - \vec{k}^2 - m^2 + i\epsilon} + \int_{2m}^{\infty} \frac{i\sigma(m')dm'}{\omega^2 - k^2 - \vec{k}^2 - m'^2 + i\epsilon} \right] B^N_{bdry}(\omega, k, \vec{k}, \dots)$$

We are interested in the action of the operator

$$\int_{-\infty}^{0} dx \int dt d\vec{r} e^{-i\omega(q,\vec{q})t} e^{i\vec{q}\vec{r}} \cos(qx) \{\partial_t^2 - \partial_x^2 - \vec{\partial}^2 + m^2 + \delta(x)\partial_x\}$$

appearing in (A.2) on (A.4). Calculating the derivatives $\partial_t, \vec{\partial}$ and performing the $t, \vec{r}$ integrals explicitely, we obtain $(2\pi)^D\delta(\omega(q,\vec{q}) - \omega)\delta(\vec{q} - \vec{k})$, and thus the ω and $\vec{k}$ integral replaces every ω by $\omega(q,\vec{q})$ and every $\vec{k}$ by $\vec{q}$. So it remains to be shown that the operator

$$\int_{-\infty}^{0} dx \cos(qx) \left\{ -\partial_x^2 - q^2 + \delta(x)\partial_x \right\}$$

operator truncates the leg

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} \left[\frac{iZ}{q^2 - k^2 + i\epsilon} + \int_{2m}^{\infty} \frac{i\sigma(m')dm'}{q'^2 - k^2 + i\epsilon} \right] B^N_{bdry}(\omega(q,\vec{q}), k, \vec{q}, \dots)$$

where $q'^2 = q^2 + m^2 - m'^2$. The actions of $-\partial_x^2 - q^2$ and $\delta(x)\partial_x$ can be computed separately. The second is simpler: it gives a factor $-ik$ and substitutes $x = 0$ in the k integral. The term $-\partial_x^2 - q^2$ gives a factor $k^2 - q^2$ but now the x integral is nontrivial and we use that the contour can be closed on the upper half plane giving

$$\begin{aligned} 2\int_{-\infty}^{0} dx \cos(qx) e^{-ikx + \epsilon x} &= \frac{1}{-i(k-q) + \epsilon} + \frac{1}{-i(k+q) + \epsilon} \\ &= i\left(\mathcal{P}_{\frac{1}{k-q}} + \mathcal{P}_{\frac{1}{k+q}}\right) + \pi\left(\delta(k-q) + \delta(k+q)\right) \end{aligned}$$

Consider first the terms containing the spectral density σ. Due to the pre-factor $k^2 - q^2$ the delta functions do not contribute and the principal value can be replaced by the function itself, which just kills the other σ term coming from the $\delta(x)\partial_x$ term. Now we analyze the terms containing Z. In the formula for $\delta(x)\partial_x$ we write

$$-ik\frac{iZ}{q^2 - k^2 + i\epsilon} = -\frac{Z}{2}\left[\frac{1}{k + q + i\epsilon} + \frac{1}{k - q - i\epsilon} \right]$$

[9] This definition of the disconnected part is slightly different from the one used for the two-point case in our previous work [21] but is more convenient for multi-point functions.

The terms with the denominator $k+q+i\epsilon$ cancel, while the ones with denominators $k-q\pm i\epsilon$ combine together to give delta functions resulting in

$$-\frac{iZ}{2}B^N_{bdry}(\omega(q,\vec{q}),q,\vec{q},\dots)$$

which, combined with the pre-factor $i\sqrt{\frac{2}{Z}}$ in the reduction formula, gives the contribution of one leg as

$$\sqrt{\frac{Z}{2}}B^N_{bdry}(\omega(q,\vec{q}),q,\vec{q},\dots)$$

Reduction of an outgoing particle gives rise to the same effect, so collecting the contributions of all legs the connected part is simply

$$\begin{aligned} R^l_k &= (2\pi)^D\delta(\sum_i\omega(p_i,\vec{p}_i)-\sum_j\omega(q_j,\vec{q}_j))\delta(\sum_i\vec{p}_i-\sum_j\vec{q}_j)\left(\frac{Z}{2}\right)^{\frac{k+l}{2}}\times \\ & B^N_{bdry}(\omega(p_i,\vec{p}_i),p_i,\vec{p}_i,\omega(q_j,\vec{q}_j),q_j,\vec{q}_j) \end{aligned} \tag{A.5}$$

In addition to the connected part treated above, there are two types of disconnected parts: one from the reduction formula itself, and the other originating from the disconnected part of the Green function (A.3). Using induction in the number of particles it is easy to show that the two contributions cancel each other, and therefore (A.5) is the final answer to the reflection factor. It also agrees with the results for the two-particle case in [21] after accounting for the difference in the normalization conventions.

A.2 Reduction formula in the closed channel

In the closed channel the Hilbert space is identical to that of the bulk theory, and the quantities of interest are the multi-particle matrix elements of the boundary state:

$$\langle B|q_1,\vec{q}_1;\dots;q_N,\vec{q}_N\rangle_{in}=(2\pi)^D\delta(\sum_j q_j)\delta(\sum_j\vec{q}_j)\omega\,(q_1,\vec{q}_1)\,K^N(\{q_i,\vec{q}_i\})$$

($\omega\,(q_1,\vec{q}_1)$ is a normalization factor introduced for convenience in order to conform with the conventions for K^1 and K^2 in the main text). In the bulk theory matrix elements are expressed in terms of the correlators via the reduction formula, but due to the presence of a boundary in time they need to be modified. First we express $A^+_{in}(q_1,\vec{q}_1)$ in terms of the *in* field as

$$\langle B|A^+_{in}(q_1,\vec{q}_1)|q_2,\vec{q}_2;\dots;q_N,\vec{q}_N\rangle_{in}=-\frac{i}{\sqrt{2}}\int_{-\infty}^{\infty}dyd\vec{r}\,e^{-i\omega(q_1,\vec{q}_1)\tau+iq_1y+i\vec{q}_1\vec{r}}\overleftrightarrow{\partial_\tau}\langle B|\Phi_{in}(\tau,y,\vec{r})|in\rangle$$

where $|in\rangle$ is a shorthand for $|q_2,\vec{q}_2;\dots;q_N,\vec{q}_N\rangle_{in}$. Now we use that for large negative τ the interacting field approaches the asymptotic field

$$\lim_{\tau\to-\infty}\Phi_{in}(\tau,y,\vec{r})=Z^{-1/2}\Phi(\tau,y,\vec{r})$$

and also

$$f(\tau)=f(0)-\int_\tau^0\partial_\tau f d\tau'$$

to obtain the rule for the elimination of a particle from the initial state:

$$\langle B|A_{in}^{+}(q_1,\vec{q}_1)|q_2,\vec{q}_2;\dots;q_N,\vec{q}_N\rangle_{in} = \frac{i}{\sqrt{2Z}}\int_{-\infty}^{\infty}dy\int_{-\infty}^{0}dt\int d\vec{r}\, e^{-i\omega(q_1,\vec{q}_1)\tau+iq_1y+i\vec{q}_1\vec{r}}$$
$$\left\{\partial_\tau^2-\partial_y^2-\vec{\partial}^2+m^2-\delta(\tau)(\partial_\tau+i\omega(q_1,\vec{q}_1))\right\}\langle B|\Phi(\tau,y,\vec{r})|in\rangle$$

Applying this procedure successively, the multi-particle matrix element can be expressed as the product of integro-differential operators

$$\frac{i}{\sqrt{2Z}}\int_{-\infty}^{\infty}dy\int_{-\infty}^{0}dt\int d\vec{r}\, e^{-i\omega(q,\vec{q})\tau+iqy+i\vec{q}\vec{r}}\left\{\partial_\tau^2-\partial_y^2-\vec{\partial}^2+m^2-\delta(\tau)(\partial_\tau+i\omega(q,\vec{q}))\right\}$$

acting on the correlator. The reduction formula obtained this way differs from its bulk counterpart by the presence of the $\delta(\tau)$ term. It also differs from the analogous expression (A.2) in the open channel by containing $e^{-i\omega(q,\vec{q})\tau}$ instead of $\cos(kx)$ and by the extra $-i\delta(\tau)\omega(q,\vec{q})$ term. Despite these differences it can be shown that it also truncates the momentum space Green function and puts the momentum on-shell. To do so we rewrite the open channel one leg contribution to the Green function (A.4) in terms of the closed channel

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega\tau}\frac{dk}{2\pi}e^{-ikx}\frac{d\vec{k}}{(2\pi)^{D-1}}e^{-i\vec{k}\vec{r}}$$
$$\left[\frac{iZ}{\omega^2-k^2-\vec{k}^2-m^2+i\epsilon}+\int_{2m}^{\infty}\frac{i\sigma(m')dm'}{\omega^2-k^2-\vec{k}^2-m'^2+i\epsilon}\right]B_{bdry}^{N}(-ik,i\omega,\vec{k},\dots)$$

In parallel with the open channel calculation we can eliminate the dependence on k, and $\vec{k}$ by performing explicitly the differentiations and integrations. Finally it remains to show that the operator

$$\int_{-\infty}^{0}d\tau\ e^{-i\omega(q,\vec{q})\tau}\left\{\partial_\tau^2+\omega(q,\vec{q})^2-\delta(t)\left(i\omega(q,\vec{q})+\partial_\tau\right)\right\}$$

truncates the leg

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega\tau}\left[\frac{iZ}{\omega^2-\omega(q,\vec{q})^2+i\epsilon}+\int_{2m}^{\infty}\frac{i\sigma(m')dm'}{\omega^2-\omega(q',\vec{q})^2+i\epsilon}\right]B_{bdry}^{N}(-iq,i\omega,\vec{q},\dots)$$

where, as before, $q'^2=q^2+m^2-m'^2$. We compute separately the contributions from $\partial_\tau^2+\omega(q,\vec{q})^2$ and $-\delta(\tau)\left(i\omega(q,\vec{q})+\partial_\tau\right)$. The second is simpler: it gives a factor $-i(\omega(q,\vec{q})+\omega)$ and substitutes $\tau=0$ in the ω integration. The operator $\partial_\tau^2+\omega(q,\vec{q})^2$ gives the factor $(-\omega^2+\omega(q,\vec{q})^2)$ and the τ integration gives

$$\int_{-\infty}^{0}d\tau\ e^{-i\omega(q,\vec{q})\tau}e^{i\omega\tau+\epsilon\tau}=\frac{-i}{\omega-\omega(q,\vec{q})-i\epsilon}=-i\mathcal{P}_{\frac{1}{\omega-\omega(q,\vec{q})}}+\pi\delta(\omega-\omega(q,\vec{q}))$$

Let us first concentrate on the σ terms. The δ function does not contribute due to the prefactor $-\omega^2+\omega(q,\vec{q})^2$, while the principal value can be replaced with the function itself and this just kills the σ term coming from the $\delta(\tau)$ part. In the Z term we use that

$$-i\frac{\omega+\omega(q,\vec{q})}{\omega^2-\omega(q,\vec{q})^2+i\epsilon}=\frac{-i}{\omega-\omega(q,\vec{q})+i\epsilon}$$

This kills the principal value term above and results in a factor two in front of the $\delta(\tau)$ part, which together gives

$$-iZB^N_{bdry}(-iq, i\omega(q,\vec{q}), \vec{q}, \dots)$$

Combining with the pre-factor $\frac{i}{\sqrt{2Z}}$ in the reduction formula gives the contribution of one leg as

$$\sqrt{\frac{Z}{2}}B^N_{bdry}(-iq, i\omega(q,\vec{q}), \vec{q}, \dots)$$

In the closed channel we have just one type of disconnected parts, the one coming from the Green functions (A.3), which contains at least one bulk two-point function as a disconnected piece. Just as in the two-particle case this does not contribute via the reduction formula, and since it appears multiplicatively the whole contribution coming from the disconnected part is vanishing. As a consequence only the connected part of the Green function contributes to the boundary state, which is therefore identical to the $q \leftrightarrow i\omega$ continuation of the result (A.5) for the reflection factor, and so K^{k+l} can be expressed in terms of R^k_l. We make this relation explicit for the two-particle term K^2 in subsection 3.3.

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